CONSERVATIVE DISCRETE TIME-INVARIANT SYSTEMS AND BLOCK OPERATOR CMV MATRICES

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Abstract. It is well known that an operator-valued function Θ from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$, where $\mathfrak{M}$ and $\mathfrak{N}$ are separable Hilbert spaces, can be realized as the transfer function of a simple conservative discrete time-invariant linear system. The known realizations involve the function Θ itself, the Hardy spaces or the reproducing kernel Hilbert spaces. On the other hand, as in the classical scalar case, the Schur class operator-valued function is uniquely determined by its so called "Schur parameters". In this paper we construct simple conservative realizations using the Schur parameters only. It turns out that the unitary operators corresponding to the systems take the form of five diagonal block operator matrices, which are the analogs of Cantero–Moral–Velázquez (CMV) matrices appeared recently in the theory of scalar orthogonal polynomials on the unit circle. We obtain new models given by truncated block operator CMV matrices for an arbitrary completely non-unitary contraction. It is shown that the minimal unitary dilations of a contraction in a Hilbert space and the minimal Naimark dilations of a semi-spectral operator measure on the unit circle can also be expressed by means of block operator CMV matrices.

Contents

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1. Introduction

In what follows the class of all continuous linear operators defined on a complex Hilbert space $\mathfrak{H}_1$ and taking values in a complex Hilbert space $\mathfrak{H}_2$ is denoted by $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $\mathbf{L}(\mathfrak{H}) := \mathbf{L}(\mathfrak{H}, \mathfrak{H})$. We denote by $I_{\mathcal{H}}$ the identity operator in a Hilbert space $\mathcal{H}$ and by $P_{\mathcal{L}}$ the orthogonal projection onto the subspace (the closed linear manifold) $\mathcal{L}$. The notation $T{\upharpoonright}\mathcal{L}$ means the restriction of a linear operator T on the set $\mathcal{L}$. The range and the null-space of a linear operator T are denoted by $\operatorname{ran} T$ and $\ker T$, respectively.

Recall that an operator $T \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is said to be

- *contractive* if $\|T\| \le 1$;
- *isometric* if $\|Tf\| = \|f\|$ for all $f \in \mathfrak{H}_1 \iff T^*T = I_{\mathfrak{H}_1}$;
- *co-isometric* if T^* is isometric $\iff TT^* = I_{\mathfrak{H}_2}$;
- *unitary* if it is both isometric and co-isometric.

Given a contraction $T \in \mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$. The operators

$$D_T := (I - T^*T)^{1/2}, \qquad D_{T^*} := (I - TT^*)^{1/2}$$

are called the *defect operators* of T, and the subspaces $\mathfrak{D}_T = \overline{\operatorname{ran}}\, D_T$, $\mathfrak{D}_{T^*} = \overline{\operatorname{ran}}\, D_{T^*}$ the *defect subspaces* of T. The dimensions $\dim \mathfrak{D}_T$, $\dim \mathfrak{D}_{T^*}$ are known as the *defect numbers* of T. The defect operators satisfy the following intertwining relations

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_T T^*. \tag{1.1}$$

It follows from (1.1) that $T\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$, $T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, and $T(\ker D_T) = \ker D_{T^*}$, $T^*(\ker D_{T^*}) = \ker D_T$. Moreover, the operators $T{\upharpoonright} \ker D_T$ and $T^*{\upharpoonright} \ker D_{T^*}$ are isometries and $T{\upharpoonright}\mathfrak{D}_T$ and $T^*{\upharpoonright}\mathfrak{D}_{T^*}$ are *pure* contractions, i.e., $||Tf|| < ||f||$ for $f \in \mathfrak{H} \setminus \{0\}$.

The *Schur class* $\mathbf{S}(\mathfrak{H}_1, \mathfrak{H}_2)$ is the set of all holomorphic and contractive $\mathbf{L}(\mathfrak{H}_1, \mathfrak{H}_2)$-valued functions on the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. This class is a natural generalization of the Schur class $\mathbf{S}$ of scalar analytic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ [56] and is intimately connected with spectral theory and models for Hilbert space contraction operators [64], [25], [26], [27], [28], [29], the Lax-Phillips scattering theory [48], [1], [23], the theory of scalar and matrix orthogonal polynomials on the unit circle $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ [37], [60], [39], [40], the theory of passive (contractive) discrete time-invariant linear systems [45], [46], [13], [14], [15], [22], [21]. One of the characterization of the operator-valued Schur class is that any $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer (characteristic) function of the form

$$\Theta(\lambda) = D + \lambda C(I_{\mathfrak{H}} - \lambda A)^{-1}B, \ \lambda \in \mathbb{D}$$

of a discrete time-invariant system (colligation)

$$\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

with the input space $\mathfrak{M}$, the output space $\mathfrak{N}$, and some state space $\mathfrak{H}$. Moreover, if the operator U_τ is given by the block operator matrix

$$U_\tau = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{array} \to \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{array},$$

then the system τ can be chosen **(a)** passive (U_τ is contractive) and minimal, **(b)** co-isometric (U_τ is co-isometry) and observable, **(c)** isometric (U_τ is isometry) and controllable, **(d)** conservative (U_τ is unitary) and simple (see Section 3). The corresponding models of the systems τ and the state space operators A are well-known. We mention the de Branges–Rovnyak functional model of a co-isometric system [26], [7], [49], the Sz.-Nagy–Foias [64], the Pavlov [52], [53], [54], and the Nikol'skiĭ–Vasyunin [50], [51] functional models of completely non-unitary contractions, the Brodskiĭ [29] functional model of a simple unitary colligation, the Arov–Kaashoek–Pik [15] functional model of a passive minimal and optimal system. All these models involve the Schur class function and/or the Hardy spaces, the de Branges–Rovnyak reproducing kernel Hilbert space.

The main goal of the present paper is constructions of models for simple conservative systems and completely non-unitary contractions by means of the operator analogs of the scalar CMV matrices, which recently appeared in the theory of orthogonal polynomials on the unit circle [31], [60], [61], [37].

In the paper of M.J. Cantero, L. Moral, and L. Velázquez [31] it is established that the semi-infinite matrices of the form

$$\mathcal{C} = \mathcal{C}(\{\alpha_n\}) = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1.2}$$

and

$$\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}(\{\alpha_n\}) = \begin{pmatrix} \bar\alpha_0 & \rho_0 & 0 & 0 & 0 & \dots \\ \bar\alpha_1\rho_0 & -\bar\alpha_1\alpha_0 & \bar\alpha_2\rho_1 & \rho_2\rho_1 & 0 & \dots \\ \rho_1\rho_0 & -\rho_1\alpha_0 & -\bar\alpha_2\alpha_1 & -\rho_2\alpha_1 & 0 & \dots \\ 0 & 0 & \bar\alpha_3\rho_2 & -\bar\alpha_3\alpha_2 & \bar\alpha_4\rho_3 & \dots \\ 0 & 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\bar\alpha_4\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1.3}$$

give representations of the unitary operator $(Uf)(\zeta) = \zeta f(\zeta)$ in $L_2(\mathbb{T}, d\mu)$, where the $d\mu$ is a nontrivial probability measure on the unite circle, with respect to the orthonormal systems obtained by orthonormalization of the sequences $\{1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \dots\}$ and $\{1, \zeta^{-1}, \zeta, \zeta^{-2}, \zeta^2, \dots\}$, respectively. The Verblunsky coefficients $\{\alpha_n\}$, $|\alpha_n| < 1$, arise in the Szegő recurrence formula

$$\zeta\Phi_n(\zeta) = \Phi_{n+1}(\zeta) + \bar\alpha_n\zeta^n\overline{\Phi_n(1/\bar\zeta)}, \qquad n = 0, 1, \dots$$

for monic orthogonal with respect to $d\mu$ polynomials $\{\Phi_n\}$, and $\rho_n := \sqrt{1-|\alpha_n|^2}$. The matrices $\mathcal{C}(\{\alpha_n\})$ $\widetilde{\mathcal{C}}(\{\alpha_n\})$ and are called the *CMV matrices.* Note that the matrix $\widetilde{C}$ is transpose to $\mathcal{C}$.

Given a probability measure μ on $\mathbb{T}$, define the *Carathéodory function* by

$$F(\lambda) = F(\lambda,\mu) := \int_{\mathbb{T}} \frac{\zeta+\lambda}{\zeta-\lambda}\, d\mu(\zeta) = 1 + 2\sum_{n=1}^{\infty} \beta_n\lambda^n, \quad \beta_n = \int_{\mathbb{T}} \zeta^{-n} d\mu$$

the moments of μ. F is an analytic function in $\mathbb{D}$ which obeys $\operatorname{Re} F > 0$, $F(0) = 1$. The Schur class function $f(\lambda)$ is then defined by

$$f(\lambda) = f(\lambda,\mu) := \frac{1}{\lambda}\,\frac{F(\lambda)-1}{F(\lambda)+1},$$

Given a Schur function $f(\lambda)$, which is not a finite Blaschke product, define inductively

$$f_0(\lambda) = f(\lambda),\ \ f_{n+1}(\lambda) = \frac{f_n(\lambda) - f_n(0)}{\lambda(1-\overline{f_n(0)}f_n(\lambda))},\ \ n \ge 0.$$

It is clear that $\{f_n\}$ is an *infinite* sequence of Schur functions called the $n-th$ *Schur iterates* and neither of its terms is a finite Blaschke product. The numbers $\gamma_n := f_n(0)$ are called the *Schur parameters:*

$$\mathcal{S}f = \{\gamma_0, \gamma_1, \ldots\}.$$

Note that

$$f_n(\lambda) = \frac{\gamma_n + \lambda f_{n+1}(\lambda)}{1+\bar\gamma_n\lambda f_{n+1}} = \gamma_n + (1-|\gamma_n|^2)\frac{\lambda f_{n+1}(\lambda)}{1+\bar\gamma_n\lambda f_{n+1}(\lambda)},\ \ n \ge 0.$$

The method of labeling $f \in \mathbf{S}$ by its Schur parameters is known as the *Schur algorithm* and is due to I. Schur [56]. In the case when

$$f(\lambda) = e^{i\varphi}\prod_{k=1}^{N}\frac{\lambda-\lambda_k}{1-\bar\lambda_k\lambda}$$

is a finite Blaschke product of order N, the Schur algorithm terminates at the N-th step. The sequence of Schur parameters $\{\gamma_n\}_{n=0}^N$ is finite, $|\gamma_n| < 1$ for $n = 0, 1, \ldots, N-1$, and $|\gamma_N| = 1$.

Due to Geronimus theorem for the function $f(\lambda,\mu)$ the relations $\gamma_n = \alpha_n$ hold true for all $n = 0, 1, \ldots$.

There is a nice multiplicative structure of the CMV matrices. In the semi-infinite case $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are the products of two matrices: $\mathcal{C} = \mathcal{L}\mathcal{M}$, $\widetilde{\mathcal{C}} = \mathcal{M}\mathcal{L}$, where

$$\begin{aligned}\mathcal{L} &= \Psi(\alpha_0)\oplus\Psi(\alpha_2)\oplus\ldots\Psi(\alpha_{2m})\oplus\ldots,\\ \mathcal{M} &= \mathbf{1}_{1\times 1}\oplus\Psi(\alpha_1)\oplus\Psi(\alpha_3)\oplus\ldots\oplus\Psi(\alpha_{2m+1})\oplus\ldots,\end{aligned}$$

and $\Psi(\alpha) = \begin{pmatrix}\bar\alpha & \rho\\ \rho & -\alpha\end{pmatrix}$. The finite $(N+1)\times(N+1)$ CMV matrices $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ obey $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{D}$ and $|\alpha_N| = 1$, and also $\mathcal{C} = \mathcal{L}\mathcal{M}$, $\widetilde{\mathcal{C}} = \mathcal{M}\mathcal{L}$, where in this case $\Psi(\alpha_N) = (\bar\alpha_N)$.

In the paper [12] it is established that the *truncated* CMV matrices

$$\mathcal{T}_0 = \mathcal{T}_0(\{\alpha_n\}) = \begin{pmatrix} -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots & \\ \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots & \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots & \\ 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$\widetilde{\mathcal{T}}_0 = \widetilde{\mathcal{T}}_0(\{\alpha_n\}) = \begin{pmatrix} -\bar\alpha_1\alpha_0 & \bar\alpha_2\rho_1 & \rho_2\rho_1 & 0 & \dots \\ -\rho_1\alpha_0 & -\bar\alpha_2\alpha_1 & -\rho_2\alpha_1 & 0 & \dots \\ 0 & \bar\alpha_3\rho_2 & -\bar\alpha_3\alpha_2 & \bar\alpha_4\rho_3 & \dots \\ 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\bar\alpha_4\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

obtained from the "full" CMV matrices $\mathcal{C} = \mathcal{C}(\{\alpha_n\})$ and $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}(\{\alpha_n\})$ by deleting the first row and the first column, provide the models of completely non-unitary contractions with rank one defect operators.

As pointed out by Simon in [61], the history of CMV matrices is started with the papers of Bunse-Gerstner and Elsner [30] (1991) and Watkins [65] (1993), where unitary semi-infinite five-diagonal matrices were introduces and studied. In [31] Cantero, Moral, and Velazquez (CMV) re-discovered them. In a context different from orthogonal polynomials on the unit circle, Bourget, Howland, and Joye [24] introduced a set of doubly infinite family of matrices with three sets of parameters which for special choices of the parameters reduces to two-sided CMV matrices on $\ell^2(\mathbb{Z})$.

The Schur algorithm for matrix valued Schur class functions and its connection with the matrix orthogonal polynomials on the unit circle have been considered in the paper of Delsarte, Genin, and Kamp [40] and in the book of Dubovoj, Fritzsche, and Kirstein [42]. The CMV matrices, connected with matrix orthogonal polynomials on the unit circle with respect to nontrivial matrix-valued measures are considered in [61], [37]. If the $k\times k$ matrix-valued non-trivial measure μ on $\mathbb{T}$, $\mu(\mathbb{T}) = I_{k\times k}$ is given, then there are the left and the right orthonormal matrix polynomials. The Szegő recursions take slightly different form than in the scalar case and the Verblunsky $k\times k$ matrix coefficients (the Schur parameters of the corresponding matrix-valued Schur function) $\{\alpha_n\}$ satisfy the inequality $||\alpha_n|| < 1$ for all n. The latter condition is in fact equivalent to the non-triviality of the measure. The entries of the corresponding CMV matrix have the size $k\times k$ and the numbers ρ_n are replaced by the $k\times k$ defect matrices $\rho_n^L = D_{\alpha_n} = (I - \alpha_n^*\alpha_n)^{1/2}$ and $\rho_n^R = D_{\alpha_n^*} = (I - \alpha_n\alpha_n^*)^{1/2}$, where α^* is the adjoint matrix. In these notations the CMV matrix is of the form [37]

$$\mathcal{C} = \mathcal{C}(\{\alpha_n\}) = \begin{pmatrix} \alpha_0^* & \rho_0^L\alpha_1^* & \rho_0^L\rho_1^L & 0 & 0 & \dots \\ \rho_0^R & -\alpha_0\alpha_1^* & -\alpha_0\rho_1^L & 0 & 0 & \dots \\ 0 & \alpha_2^*\rho_1^R & -\alpha_2^*\alpha_1 & \rho_2^L\alpha_3^* & \rho_2^L\rho_3^L & \dots \\ 0 & \rho_2^R\rho_1^R & -\rho_2^R\alpha_1 & -\alpha_2\alpha_3^* & -\alpha_2\rho_3^L & \dots \\ 0 & 0 & 0 & \alpha_4^*\rho_3^R & -\alpha_4^*\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{1.4}$$

The operator extension of the Schur algorithm was developed by T. Constantinescu in [34] and with numerous applications is presented in the monographs [20], [36]. The next theorem

goes back to Shmul'yan [57], [58] and T. Constantinescu [34] (see also [20], [8], [9]) and plays a key role in the operator Schur algorithm.

Theorem 1.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be separable Hilbert spaces and let the function $\Theta(\lambda)$ be from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$. Then there exists a function $Z(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)})$ such that*

$$\Theta(\lambda)=\Theta(0)+D_{\Theta^*(0)}Z(\lambda)(I+\Theta^*(0)Z(\lambda))^{-1}D_{\Theta(0)},\ \lambda\in\mathbb{D}. \tag{1.5}$$

The representation (1.5) of a function $\Theta(\lambda)$ from the Schur class is called the Möbius representation of $\Theta(\lambda)$ and the function $Z(\lambda)$ is called the Möbius parameter of $\Theta(\lambda)$ (see [8], [9]). Clearly, $Z(0)=0$ and by Schwartz's lemma we obtain that

$$||Z(\lambda)||\le|\lambda|,\ \lambda\in\mathbb{D}.$$

The operator Schur's algorithm [20]. Fix $\Theta(\lambda)\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$, put $\Theta_0(\lambda)=\Theta(\lambda)$ and let $Z_0(\lambda)$ be the Möbius parameter of Θ. Define

$$\Gamma_0=\Theta(0),\ \Theta_1(\lambda)=\frac{Z_0(\lambda)}{\lambda}\in\mathbf{S}(\mathfrak{D}_{\Gamma_0},\mathfrak{D}_{\Gamma_0^*}),\ \Gamma_1=\Theta_1(0)=Z_0'(0).$$

If $\Theta_0(\lambda),\ldots,\Theta_n(\lambda)$ and $\Gamma_0,\ldots,\Gamma_n$ have been chosen, then let $Z_{n+1}(\lambda)\in\mathbf{S}(\mathfrak{D}_{\Gamma_n},\mathfrak{D}_{\Gamma_n^*})$ be the Möbius parameter of Θ_n. Put

$$\Theta_{n+1}(\lambda)=\frac{Z_{n+1}(\lambda)}{\lambda},\ \Gamma_{n+1}=\Theta_{n+1}(0).$$

The contractions $\Gamma_0\in\mathbf{L}(\mathfrak{M},\mathfrak{N})$, $\Gamma_n\in\mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}})$, $n=1,2,\ldots$ are called the *Schur parameters* of $\Theta(\lambda)$ and the function $\Theta_n(\lambda)\in\mathbf{S}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}})$ we will call the *$n-th$ Schur iterate* of $\Theta(\lambda)$.

Formally we have

$$\Theta_{n+1}(\lambda)\upharpoonright\operatorname{ran}D_{\Gamma_n}=\frac{1}{\lambda}D_{\Gamma_n^*}(I_{\mathfrak{D}_{\Gamma_n^*}}-\Theta_n(\lambda)\Gamma_n^*)^{-1}(\Theta_n(\lambda)-\Gamma_n)D^{-1}_{\Gamma_n}\upharpoonright\operatorname{ran}D_{\Gamma_n}.$$

Clearly, the sequence of Schur parameters $\{\Gamma_n\}$ is infinite if and only if all operators Γ_n are non-unitary. The sequence of Schur parameters consists of a finite number of operators Γ_0, $\Gamma_1,\ldots,\Gamma_N$ if and only if $\Gamma_N\in\mathbf{L}(\mathfrak{D}_{\Gamma_{N-1}},\mathfrak{D}_{\Gamma^*_{N-1}})$ is unitary. If Γ_N is isometric (co-isometric) then $\Gamma_n=0$ for all $n>N$. The following generalization of the classical Schur result is proved in [34] (see also [20]).

Theorem 1.2. *There is a one-to-one correspondence between the Schur class functions $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ and the set of all sequences of contractions $\{\Gamma_n\}_{n\ge0}$ such that*

$$\Gamma_0\in\mathbf{L}(\mathfrak{M},\mathfrak{N}),\ \Gamma_n\in\mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}}),\ n\ge1. \tag{1.6}$$

A sequence of contractions of the form (1.6) is called the *choice sequence* [32]. Such objects are used for the indexing of contractive intertwining dilations, of positive Toeplitz forms, and of the Naimark dilations of semi-spectral measures on the unit circle (see [32], [33], [35], [20], [36]). Observe that the Naimark dilation and the model of a simple conservative system are given in [33], [34], and [20] by infinite in all sides block operator matrix whose entries are expressed by means of the choice sequence or the Schur parameters.

Let us describe the main results of our paper. Given a choice sequence (1.6). We construct the Hilbert spaces $\mathfrak{H}_0 = \mathfrak{H}_0(\{\Gamma_n\}_{n\ge 0})$, $\widetilde{\mathfrak{H}}_0 = \widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\ge 0})$, the unitary operators

$$\mathcal{U}_0 = \mathcal{U}_0(\{\Gamma_n\}_{n\ge 0}) = \begin{bmatrix}\Gamma_0 & \mathcal{G}_0\\ \mathcal{F}_0 & \mathcal{T}_0\end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{H}_0\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus\\ \mathfrak{H}_0\end{array},\quad \widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 0}) = \begin{bmatrix}\Gamma_0 & \widetilde{\mathcal{G}}_0\\ \widetilde{\mathcal{F}}_0 & \widetilde{\mathcal{T}}_0\end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus\\ \widetilde{\mathfrak{H}}_0\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus\\ \widetilde{\mathfrak{H}}_0\end{array},$$

and the unitarily equivalent simple conservative systems

$$\zeta_0 = \left\{\begin{bmatrix}\Gamma_0 & \mathcal{G}_0\\ \mathcal{F}_0 & \mathcal{T}_0\end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_0\right\},\quad \widetilde{\zeta}_0 = \left\{\begin{bmatrix}\Gamma_0 & \widetilde{\mathcal{G}}_0\\ \widetilde{\mathcal{F}}_0 & \widetilde{\mathcal{T}}_0\end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \widetilde{\mathfrak{H}}_0\right\},$$

such that the Schur parameters of the transfer function Θ of the systems ζ_0 and $\widetilde{\zeta}_0$ are precisely $\{\Gamma_n\}_{n\ge 0}$. Moreover, the operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ in such constructions are given by the operator analogs of the CMV matrices. In the case when the operators Γ_n are neither isometric nor co-isometric for each $n = 0, 1, \ldots$, the Hilbert spaces $\mathfrak{H}_0$ and $\widetilde{\mathfrak{H}}_0$ are of the form

$$\mathfrak{H}_0 = \sum_{n\ge 0}\bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array},\quad \widetilde{\mathfrak{H}}_0 = \sum_{n\ge 0}\bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{array},$$

and the operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ are given by the products of unitary diagonal operator matrices

$$\mathcal{U}_0 = \begin{bmatrix}\mathbf{J}_{\Gamma_0} & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \mathbf{J}_{\Gamma_4} & \\ & & & \ddots\end{bmatrix}\begin{bmatrix}I_{\mathfrak{M}} & & & \\ & \mathbf{J}_{\Gamma_1} & & \\ & & \mathbf{J}_{\Gamma_3} & \\ & & & \ddots\end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \begin{bmatrix}I_{\mathfrak{N}} & & & \\ & \mathbf{J}_{\Gamma_1} & & \\ & & \mathbf{J}_{\Gamma_3} & \\ & & & \ddots\end{bmatrix}\begin{bmatrix}\mathbf{J}_{\Gamma_0} & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \mathbf{J}_{\Gamma_4} & \\ & & & \ddots\end{bmatrix},$$

where

$$\mathbf{J}_{\Gamma_0} = \begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}\\ D_{\Gamma_0} & -\Gamma_0^*\end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_{\Gamma_0^*}\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus\\ \mathfrak{D}_{\Gamma_0}\end{array},\quad \mathbf{J}_{\Gamma_k} = \begin{bmatrix}\Gamma_k & D_{\Gamma_k^*}\\ D_{\Gamma_k} & -\Gamma_k^*\end{bmatrix} : \begin{array}{c}\mathfrak{D}_{\Gamma_{k-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_k^*}\end{array} \to \begin{array}{c}\mathfrak{D}_{\Gamma^*_{k-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_k}\end{array},\quad k = 1, 2, \ldots$$

are the unitary operators called "elementary rotations" [20]. The operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ take the form of five-diagonal block operator matrices

$$\mathcal{U}_0 = \begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 & D_{\Gamma_2^*}D_{\Gamma_3^*} & 0 & 0 & 0 & \ldots\\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 & -\Gamma_2^*D_{\Gamma_3^*} & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & \Gamma_4 D_{\Gamma_3} & -\Gamma_4\Gamma_3^* & D_{\Gamma_4^*}\Gamma_5 & D_{\Gamma_4^*}D_{\Gamma_5^*} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}$$

and

$$\widetilde{\mathcal{U}}_0=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*} & 0 & 0 & 0 & \ldots\\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & -\Gamma_1^* D_{\Gamma_2^*} & 0 & 0 & 0 & \ldots\\ 0 & 0 & \Gamma_3 D_{\Gamma_2} & -\Gamma_3\Gamma_2^* & D_{\Gamma_3^*}\Gamma_4 & D_{\Gamma_3^*}D_{\Gamma_4^*} & 0 & \ldots\\ 0 & 0 & D_{\Gamma_3}D_{\Gamma_2} & -D_{\Gamma_3}\Gamma_2^* & -\Gamma_3^*\Gamma_4 & -\Gamma_3^* D_{\Gamma_4^*} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}$$

Note that the following relation

$$\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 0})=(\mathcal{U}_0(\{\Gamma_n^*\}_{n\ge 0})^*.$$

holds true. Hence the CMV matrix (1.4) corresponds to the case

$$\begin{aligned}&\mathfrak{M}=\mathfrak{N}=\mathfrak{D}_{\Gamma_0}=\mathfrak{D}_{\Gamma_0^*}=\mathfrak{D}_{\Gamma_1}=\mathfrak{D}_{\Gamma_1^*}=\ldots=\mathfrak{D}_{\Gamma_n}=\mathfrak{D}_{\Gamma_n^*}=\ldots=\mathbb{C}^k,\\ &\alpha_n=\Gamma_n^*,\ n=0,1,\ldots,\end{aligned}$$

Thus,

$$\mathcal{C}(\{\alpha_n\})=\mathcal{U}_0(\{\Gamma_n^*\}_{n\ge0})=\left(\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge0}\right)^*,\ \widetilde{\mathcal{C}}(\{\alpha_n\})=\widetilde{\mathcal{U}}_0(\{\Gamma_n^*\}_{n\ge0})=(\mathcal{U}_0(\{\Gamma_n\}_{n\ge0})^*.$$

The block operator truncated CMV matrices

$$\mathcal{T}_0=\mathcal{T}_0(\{\Gamma_n\}_{n\ge0}):=P_{\mathfrak{H}_0}\mathcal{U}_0{\upharpoonright}\mathfrak{H}_0\ \text{ and } \widetilde{\mathcal{T}}_0=\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\ge0}):=P_{\widetilde{\mathfrak{H}}_0}\widetilde{\mathcal{U}}_0{\upharpoonright}\widetilde{\mathfrak{H}}_0$$

are given by

$$\mathcal{T}_0=\begin{bmatrix}-\Gamma_0^* & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \mathbf{J}_{\Gamma_4} & \\ & & & \ddots\end{bmatrix}\begin{bmatrix}\mathbf{J}_{\Gamma_1} & & \\ & \mathbf{J}_{\Gamma_3} & \\ & & \ddots\end{bmatrix},\ \widetilde{\mathcal{T}}_0=\begin{bmatrix}\mathbf{J}_{\Gamma_1} & & \\ & \mathbf{J}_{\Gamma_3} & \\ & & \ddots\end{bmatrix}\begin{bmatrix}-\Gamma_0^* & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \mathbf{J}_{\Gamma_4} & \\ & & & \ddots\end{bmatrix},$$

and can be rewritten in the three diagonal block operator matrix form with 2×2 entries

$$\mathcal{T}_0=\begin{bmatrix}\mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & 0 & \cdot\\ \mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & 0 & \cdot\\ 0 & \mathcal{A}_2 & \mathcal{B}_3 & \mathcal{C}_3 & 0 & \cdot\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix},\ \widetilde{\mathcal{T}}_0=\begin{bmatrix}\widetilde{\mathcal{B}}_1 & \widetilde{\mathcal{C}}_1 & 0 & 0 & 0 & \cdot\\ \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{B}}_2 & \widetilde{\mathcal{C}}_2 & 0 & 0 & \cdot\\ 0 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathcal{B}}_3 & \widetilde{\mathcal{C}}_3 & 0 & \cdot\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

The constructions above and the corresponding results are presented in Section 5. We essentially rely on the constructions of simple conservative realizations of the Schur iterates $\{\Theta_n(\lambda)\}_{n\ge1}$ by means of a given simple conservative realization of the function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ [9]. A brief survey of the results in [9] are given in Section 4. The cases when the Schur parameter $\Gamma_m\in\mathbf{L}(\mathfrak{D}_{\Gamma_{m-1}},\mathfrak{D}_{\Gamma^*_{m-1}})$ of the function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$, is isometric, co-isometric, unitary are considered in detail in Section 6. Observe that in fact we give another prove of Theorem 1.2 (the uniqueness of the function from $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ with with given its Schur parameters is proved in Section 2). In Section 7 we obtain in the block operator CMV matrix form the minimal unitary dilations of a contraction and the minimal Naimark dilations of a semi-spectral measure on the unite circle. Another and more complicated constructions of the minimal Naimark dilation and a simple conservative realization for a function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ by means of its Schur parameters are given in [33] and in [47], respectively (see also [20]).

Simple conservative realizations of scalar Schur functions with operators A, B, C, and D expressed via corresponding Schur parameters have been obtained by V. Duboboj [41].

We also prove in Section 7 that a unitary operator U in a separable Hilbert space $\mathfrak{K}$ having a cyclic subspace $\mathfrak{M}$ ($\overline{\operatorname{span}}\,\{U^n\mathfrak{M},\ n\in\mathbb{Z}\}=\mathfrak{K}$) is unitarily equivalent to the block operator CMV matrices $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ constructed by means of the Schur parameters of the function $\Theta(\lambda)=\dfrac{1}{\lambda}(F^*_{\mathfrak{M}}(\bar\lambda)-I_{\mathfrak{M}})(F^*_{\mathfrak{M}}(\bar\lambda)+I_{\mathfrak{M}})^{-1}$, where $F_{\mathfrak{M}}(\lambda)=P_{\mathfrak{M}}(U+\lambda I_{\mathfrak{K}})(U-\lambda I_{\mathfrak{K}})^{-1}\upharpoonright\mathfrak{M}$, $\lambda\in\mathbb{D}$.

In the last Section 8 we prove that the Sz.-Nagy–Foias [64] characteristic functions of truncated block operator CMV matrices $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$, constructing by means of the Schur parameters $\{\Gamma_n\}_{n\ge 0}$ of a purely contractive function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$, coincide with Θ in the sense of [64].

2. The Schur class functions and their iterates

In the sequel we need the well known fact [64], [20] that if $T\in\mathbf{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ is a contraction which is neither isometric nor co-isometric, then the operator (*elementary rotation* [20]) $\mathbf{J}_T$ given by the operator matrix

$$\mathbf{J}_T=\begin{bmatrix}T & D_{T^*}\\ D_T & -T^*\end{bmatrix}:\ \begin{array}{c}\mathfrak{H}_1\\ \oplus\\ \mathfrak{D}_{T^*}\end{array}\ \to\ \begin{array}{c}\mathfrak{H}_2\\ \oplus\\ \mathfrak{D}_T\end{array}$$

is unitary. Clearly, $\mathbf{J}^*_T=\mathbf{J}_{T^*}$. If T is isometric or co-isometric, then the corresponding unitary elementary rotation takes the row or the column form

$$\mathbf{J}^{(r)}_T=\begin{bmatrix}T & I_{\mathfrak{D}_{T^*}}\end{bmatrix}:\ \begin{array}{c}\mathfrak{H}_1\\ \oplus\\ \mathfrak{D}_{T^*}\end{array}\ \to\mathfrak{H}_2,\ \mathbf{J}^{(c)}_T=\begin{bmatrix}T\\ D_T\end{bmatrix}:\mathfrak{H}_1\to\ \begin{array}{c}\mathfrak{H}_2\\ \oplus\\ \mathfrak{D}_T\end{array},$$

and

$$\left(\mathbf{J}^{(r)}_T\right)^*=\mathbf{J}^{(c)}_{T^*}.$$

In Section 5 we will need the following statement.

Proposition 2.1. [11]. *Let T be a contraction. Then $Th=D_{T^*}g$ if and only if there exists a vector $\varphi\in\mathfrak{D}_T$ such that $h=D_T\varphi$ and $g=T\varphi$.*

Recall that if $\Theta(\lambda)\in\mathbf{S}(\mathfrak{H}_1,\mathfrak{H}_2)$ then there is a uniquely determined decomposition [64, Proposition V.2.1]

$$\Theta(\lambda)=\begin{bmatrix}\Theta_p(\lambda) & 0\\ 0 & \Theta_u\end{bmatrix}:\ \begin{array}{c}\mathfrak{D}_{\Theta(0)}\\ \oplus\\ \ker D_{\Theta(0)}\end{array}\ \to\ \begin{array}{c}\mathfrak{D}_{\Theta^*(0)}\\ \oplus\\ \ker D_{\Theta^*(0)}\end{array},$$

where $\Theta_p(\lambda)\in\mathbf{S}(\mathfrak{D}_{\Theta(0)},\mathfrak{D}_{\Theta^*(0)})$, $\Theta_p(0)$ is a pure contraction and Θ_u is a unitary constant. The function $\Theta_p(\lambda)$ is called the *pure part* of $\Theta(\lambda)$ (see [20]). If $\Theta(0)$ is isometric (respect., co-isometric) then the pure part is of the form $\Theta_p(\lambda)=0\in\mathbf{S}(\{0\},\mathfrak{D}_{\Theta^*(0)})$ (respect., $\Theta_p(\lambda)=0\in\mathbf{S}(\mathfrak{D}_{\Theta(0)},\{0\})$). The function Θ is called purely contractive if $\ker D_{\Theta(0)}=\{0\}$. Two operator-valued functions $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ and $\Omega\in\mathbf{S}(\mathfrak{K},\mathfrak{L})$ *coincide* [64] if there are two unitary operators $V:\mathfrak{N}\to\mathfrak{L}$ and $U:\mathfrak{K}\to\mathfrak{M}$ such that

$$V\Theta(\lambda)U=\Omega(\lambda),\quad \lambda\in\mathbb{D}. \tag{2.1}$$

For the corresponding Schur parameters and the Schur iterates relation (2.1) yields the equalities

$$(2.2)\qquad \begin{array}{l} G_n = V\Gamma_n U,\\ \mathfrak{D}_{G_n} = U^*\mathfrak{D}_{\Gamma_n},\ \mathfrak{D}_{G_n^*} = V\mathfrak{D}_{\Gamma_n^*},\ D_{G_n} = U^* D_{\Gamma_n} U,\ D_{G_n^*} = V D_{\Gamma_n^*} V^*,\\ V\Theta_n(\lambda)U = \Omega_n(\lambda),\ \lambda\in\mathbb{D} \end{array}$$

for all $n = 0, 1, \ldots$.

In what follows we give a proof of Theorem 1.6 different from the original one in [34]. First of all we will prove the uniqueness. The existence will be proved in Section 5.

Theorem 2.2. *Any choice sequence uniquely determines a Schur class function.*

Proof. Let $\Gamma_0 \in \mathbf{L}(\mathfrak{M},\mathfrak{N})$, $\Gamma_n \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma^*_{n-1}})$, $n \geq 1$ be a choice sequence. Suppose the functions $\Theta_0(\lambda)$ and $\widehat{\Theta}_0(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ have $\{\Gamma_n\}_0^\infty$ as their Schur parameters. Then for every $n = 0, 1, \ldots$ hold the relations

$$\begin{array}{l} \Theta_n(\lambda) = \Gamma_n + \lambda D_{\Gamma_n^*}(I + \lambda\Theta_{n+1}(\lambda)\Gamma_n^*)^{-1}\Theta_{n+1}(\lambda)D_{\Gamma_n},\\ \widehat{\Theta}_n(\lambda) = \Gamma_n + \lambda D_{\Gamma_n^*}(I + \lambda\widehat{\Theta}_{n+1}(\lambda)\Gamma_n^*)^{-1}\widehat{\Theta}_{n+1}(\lambda)D_{\Gamma_n}, \end{array}$$

where $\{\Theta_n(\lambda)\}$ and $\{\widehat{\Theta}_n(\lambda)\}$ are the Schur iterates of Θ and $\widehat{\Theta}$, respectively. Then one has for every n the equalities

(2.3)

$$\begin{array}{l} \Theta_n(\lambda) - \widehat{\Theta}_n(\lambda) =\\ \quad = \lambda D_{\Gamma_n^*}(I + \lambda\Theta_{n+1}(\lambda)\Gamma_n^*)^{-1}(\Theta_{n+1}(\lambda) - \widehat{\Theta}_{n+1}(\lambda))(I + \lambda\widehat{\Theta}_{n+1}(\lambda)\Gamma_n^*)^{-1}D_{\Gamma_n},\ \lambda\in\mathbb{D}. \end{array}$$

Since $||\Theta_{n+1}(\lambda) - \widehat{\Theta}_{n+1}(\lambda)|| \leq 2$ for all $\lambda\in\mathbb{D}$ and $\Theta_{n+1}(0) = \widehat{\Theta}_{n+1}(0) = \Gamma_{n+1}$, by Schwartz's lemma we get

$$||\Theta_{n+1}(\lambda) - \widehat{\Theta}_{n+1}(\lambda)|| \leq 2\,|\lambda|,\ \lambda\in\mathbb{D}.$$

Further

$$\begin{array}{l} ||(I + \lambda\Theta_{n+1}(\lambda)\Gamma_n^*)f|| \geq (1 - |\lambda|)||f||,\\ ||(I + \lambda\widehat{\Theta}_{n+1}(\lambda)\Gamma_n^*)f|| \geq (1 - |\lambda|)||f|| \end{array}$$

for all $\lambda\in\mathbb{D}$ and for all $f \in \mathfrak{D}_{\Gamma^*_{n-1}}$. These relations imply

$$||(I + \lambda\Theta_{n+1}(\lambda)\Gamma_n^*)^{-1}|| \leq \frac{1}{1-|\lambda|},\ ||(I + \lambda\widehat{\Theta}_{n+1}(\lambda)\Gamma_n^*)^{-1}|| \leq \frac{1}{1-|\lambda|}$$

for all $\lambda\in\mathbb{D}$ and for all $n = 0, 1, \ldots$. Hence and from (2.3) we have

$$||\Theta_n(\lambda) - \widehat{\Theta}_n(\lambda)|| \leq 2|\lambda|\,\frac{|\lambda|}{(1-|\lambda|)^2},\ \lambda\in\mathbb{D}.$$

Then applying (2.3) for Θ_{n-1} and $\widehat{\Theta}_{n-1}$ in the left hand side, we see that

$$||\Theta_{n-1}(\lambda) - \widehat{\Theta}_{n-1}(\lambda)|| \leq 2|\lambda|\left(\frac{|\lambda|}{(1-|\lambda|)^2}\right)^2,\ \lambda\in\mathbb{D},$$

and finally

$$(2.4)\qquad ||\Theta_0(\lambda) - \widehat{\Theta}_0(\lambda)|| \leq 2|\lambda|\left(\frac{|\lambda|}{(1-|\lambda|)^2}\right)^{n+1},\ \lambda\in\mathbb{D}$$

for all $n = 0, 1, \ldots$.

Let $|\lambda| < (3-\sqrt{5})/2$. Then

$$\frac{|\lambda|}{(1-|\lambda|)^2} < 1.$$

Letting $n \to \infty$ in (2.4) we get

$$\Theta_0(\lambda) = \widehat{\Theta}_0(\lambda),\ |\lambda| < \frac{3-\sqrt{5}}{2}.$$

Since Θ_0 and $\widehat{\Theta}_0$ are holomorphic in $\mathbb{D}$, they are equal on $\mathbb{D}$. $\square$

3. Conservative discrete-time linear systems and their transfer functions

Let $\mathfrak{M}, \mathfrak{N}$, and $\mathfrak{H}$ be separable Hilbert spaces. A linear system $\tau = \left\{\begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}\right\}$ with bounded linear operators A, B, C, D of the form

$$\text{(3.1)} \qquad \begin{cases} \sigma_k = Ch_k + D\xi_k, \\ h_{k+1} = Ah_k + B\xi_k \end{cases} \qquad k \ge 0,$$

where $\{\xi_k\} \subset \mathfrak{M}$, $\{\sigma_k\} \subset \mathfrak{N}$, $\{h_k\} \subset \mathfrak{H}$ is called a *discrete time-invariant system*. The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and the Hilbert space $\mathfrak{H}$ is called the state space. The operators A, B, C, and D are called the state space operator, the control operator, the observation operator, and the feedthrough operator of τ, respectively. Put

$$U_\tau = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{array} \to \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H} \end{array}$$

If U_τ is contractive, then the corresponding discrete-time system is said to be *passive* [13]. If the operator U_τ is isometric (respect., co-isometric, unitary), then the system is said to be *isometric* (respect., *co-isometric*, *conservative*). Isometric, co-isometric, conservative, and passive discrete time-invariant systems have been studied in [25], [26], [7], [64], [45], [46], [27], [29], [22], [6], [13], [14], [15], [16], [17], [18], [62], [63], [10], [8], [9], [43]. It is relevant to remark that a brief history of System Theory is presented in the recent preprint of B. Fritzsche, V. Katsnelson, and B. Kirstein [43].

The subspaces

$$\text{(3.2)} \qquad \mathfrak{H}^c := \overline{\operatorname{span}}\,\{A^nB\mathfrak{M} :\ n = 0,1,\dots\} \text{ and } \mathfrak{H}^o := \overline{\operatorname{span}}\,\{A^{*n}C^*\mathfrak{N} :\ n = 0,1,\dots\}$$

are said to be the *controllable* and *observable* subspaces of the system τ, respectively. The system τ is said to be *controllable* (respect., *observable*) if $\mathfrak{H}^c = \mathfrak{H}$ (respect., $\mathfrak{H}^o = \mathfrak{H}$), and it is called *minimal* if τ is both controllable and observable. The system τ is said to be *simple* if

$$\mathfrak{H} = \operatorname{clos}\{\mathfrak{H}^c + \mathfrak{H}^o\} = \overline{\operatorname{span}}\,\{A^kB\mathfrak{M},\ A^{*l}C^*\mathfrak{N},\ k,l = 0,1,\dots\}$$

It follows from (3.2) that

$$(\mathfrak{H}^c)^\perp = \bigcap_{n=0}^{\infty} \ker(B^*A^{*n}), \quad (\mathfrak{H}^o)^\perp = \bigcap_{n=0}^{\infty} \ker(CA^n),$$

and therefore there are the following alternative characterizations:

(a) τ is controllable $\iff \bigcap\limits_{n=0}^{\infty}\ker(B^*A^{*n})=\{0\}$;

(b) τ is observable $\iff \bigcap\limits_{n=0}^{\infty}\ker(CA^{n})=\{0\}$;

(c) τ is simple $\iff \left(\bigcap\limits_{n=0}^{\infty}\ker(B^*A^{*n})\right)\cap\left(\bigcap\limits_{n=0}^{\infty}\ker(CA^{n})\right)=\{0\}$.

A contraction A acting in a Hilbert space $\mathfrak{H}$ is called *completely non-unitary* [64] if there is no nontrivial reducing subspace of A, on which A generates a unitary operator. Given a contraction A in $\mathfrak{H}$ then there is a canonical orthogonal decomposition [64, Theorem I.3.2]

$$\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1,\qquad A=A_0\oplus A_1,\quad A_j=A{\upharpoonright}\mathfrak{H}_j,\quad j=0,1,$$

where $\mathfrak{H}_0$ and $\mathfrak{H}_1$ reduce A, the operator A_0 is a completely non-unitary contraction, and A_1 is a unitary operator. Moreover,

$$\mathfrak{H}_1=\left(\bigcap_{n\ge1}\ker D_{A^n}\right)\bigcap\left(\bigcap_{n\ge1}\ker D_{A^{*n}}\right).$$

Since

$$\bigcap_{k=0}^{n-1}\ker(D_AA^k)=\ker D_{A^n},\ \bigcap_{k=0}^{n-1}\ker(D_{A^*}A^{*k})=\ker D_{A^{*n}},$$

we get

$$\begin{aligned}&\bigcap_{n\ge1}\ker D_{A^n}=\mathfrak{H}\ominus\overline{\rm span}\,\left\{A^{*n}D_A\mathfrak{H},\ n=0,1,\ldots\right\},\\ &\bigcap_{n\ge1}\ker D_{A^{*n}}=\mathfrak{H}\ominus\overline{\rm span}\,\left\{A^{n}D_{A^*}\mathfrak{H},\ n=0,1,\ldots\right\}.\end{aligned}\tag{3.3}$$

It follows that

$$\begin{aligned}&A\text{ is completely non-unitary }\iff\left(\bigcap_{n\ge1}\ker D_{A^n}\right)\bigcap\left(\bigcap_{n\ge1}\ker D_{A^{*n}}\right)=\{0\}\iff\\ &\iff\overline{\rm span}\,\{A^{*n}D_A,\ A^mD_{A^*},\ n,m\ge0\}=\mathfrak{H}.\end{aligned}\tag{3.4}$$

If $\tau=\left\{\begin{bmatrix}D&C\\B&A\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{H}\right\}$ is a conservative system then τ is simple if and only if the state space operator A is a completely non-unitary contraction [29], [22].

The *transfer function*

$$\Theta_\tau(\lambda):=D+\lambda C(I_{\mathfrak{H}}-\lambda A)^{-1}B,\quad\lambda\in\mathbb{D},$$

of the passive system τ belongs to the Schur class $\mathbf{S}(\mathfrak{M},\mathfrak{N})$ [13]. Conservative systems are also called the unitary colligations and their transfer functions are called the characteristic functions [29].

The examples of conservative systems are given by

$$\Sigma=\left\{\begin{bmatrix}-A&D_{A^*}\\D_A&A^*\end{bmatrix};\mathfrak{D}_A,\mathfrak{D}_{A^*},\mathfrak{H}\right\},\ \Sigma_*=\left\{\begin{bmatrix}-A^*&D_A\\D_{A^*}&A\end{bmatrix};\mathfrak{D}_{A^*},\mathfrak{D}_A,\mathfrak{H}\right\}.$$

The transfer functions of these systems

$$\Phi_\Sigma(\lambda)=\left(-A+\lambda D_{A^*}(I_{\mathfrak{H}}-\lambda A^*)^{-1}D_A\right){\upharpoonright}\mathfrak{D}_A,\quad\lambda\in\mathbb{D}$$

and

$$\Phi_{\Sigma_*}(\lambda) = \left(-A^* + \lambda D_A(I_{\mathfrak{H}} - \lambda A)^{-1} D_{A^*}\right) \upharpoonright \mathfrak{D}_{A^*}, \quad \lambda \in \mathbb{D}$$

are precisely the Sz.Nagy–Foias characteristic functions [64] of A and A^*, correspondingly.

It is well known that every operator-valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer function of some passive system, which can be chosen as controllable isometric (respect., observable co-isometric, simple conservative, minimal passive); cf. [26], [64], [29], [7] [13], [15], [6]. Moreover, two controllable isometric (respect., observable co-isometric, simple conservative) systems with the same transfer function are *unitarily equivalent*: two discrete-time systems

$$\tau_1 = \left\{ \begin{bmatrix} D & C_1 \\ B_1 & A_1 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_1 \right\} \quad \text{and} \quad \tau_2 = \left\{ \begin{bmatrix} D & C_2 \\ B_2 & A_2 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_2 \right\}$$

are said to be unitarily equivalent if there exists a unitary operator V from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$\begin{gathered} A_1 = V^{-1} A_2 V, \quad B_1 = V^{-1} B_2, \quad C_1 = C_2 V \iff \\ \iff \begin{bmatrix} I_{\mathfrak{N}} & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} D & C_1 \\ B_1 & A_1 \end{bmatrix} = \begin{bmatrix} D & C_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & V \end{bmatrix} \end{gathered} \tag{3.5}$$

cf. [25], [26], [7], [29], [6].

4. Conservative realizations of the Schur iterates

Let A be a completely non-unitary contraction in a separable Hilbert space $\mathfrak{H}$. Suppose $\ker D_A \neq \{0\}$. Define the subspaces and operators (see [9])

$$\left\{ \begin{array}{l} \mathfrak{H}_{0,0} := \mathfrak{H} \\ \mathfrak{H}_{n,0} = \ker D_{A^n},\ \mathfrak{H}_{0,m} := \ker D_{A^{*m}}, \\ \mathfrak{H}_{n,m} := \ker D_{A^n} \cap \ker D_{A^{*m}},\ m, n \in \mathbb{N}, \end{array} \right. \tag{4.1}$$

$$A_{n,m} := P_{n,m} A \upharpoonright \mathfrak{H}_{n,m} \in \mathbf{L}(\mathfrak{H}_{n,m}), \tag{4.2}$$

where $P_{n,m}$ are the orthogonal projections in $\mathfrak{H}$ onto $\mathfrak{H}_{n,m}$. The next results have been established in [9].

Theorem 4.1. [9].

(1) *Hold the relations*

$$\ker D_{A^k_{n,m}} = \mathfrak{H}_{n+k,m},\ \ker D_{A^{*k}_{n,m}} = \mathfrak{H}_{n,m+k},\ k = 1, 2, \ldots,$$

$$\left\{ \begin{array}{l} \mathfrak{D}_{A_{n,m}} = \overline{\operatorname{ran}}\,(P_{n,m} D_{A^{n+1}}), \\ \mathfrak{D}_{A^*_{n,m}} = \overline{\operatorname{ran}}\,(P_{n,m} D_{A^{*m+1}}) \end{array} \right. ,$$

$$\left\{ \begin{array}{l} A\mathfrak{H}_{n,m} = \mathfrak{H}_{n-1,m+1},\ n \geq 1, \\ A^*\mathfrak{H}_{n,m} = \mathfrak{H}_{n+1,m-1},\ m \geq 1 \end{array} \right. ,$$

$$\left(A_{n,m}\right)_{k,l} = A_{n+k,m+l}. \tag{4.3}$$

(2) *The operators $\{A_{n,m}\}$ are completely non-unitary contractions.*

(3) *The operators*

$$A_{n,0},\ A_{n-1,1},\ \ldots, A_{n-k,k}, \ldots, A_{0,n}$$

are unitarily equivalent and

$$A_{n-1,m+1}Af = AA_{n,m}f,\ f \in \mathfrak{H}_{n,m},\ n \geq 1.$$

The relation (4.3) yields the following picture for the creation of the operators $A_{n,m}$:

$$\begin{tikzcd}[column sep=small]
 & & & A \arrow[dl] \arrow[dr] & & & \\
 & & A_{1,0} \arrow[dl] \arrow[dr] & & A_{0,1} \arrow[dl] \arrow[dr] & & \\
 & A_{2,0} \arrow[dl] \arrow[dr] & & A_{1,1} \arrow[dl] \arrow[dr] & & A_{0,2} \arrow[dl] \arrow[dr] & \\
A_{3,0} & & A_{2,1} & & A_{1,2} & & A_{0,3} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{tikzcd}$$

The process terminates on the N-th step if and only if

$$\begin{aligned}
&\ker D_{A^N} = \{0\} \iff \ker D_{A^{N-1}} \cap \ker D_{A^*} = \{0\} \iff \ldots \\
&\ker D_{A^{N-k}} \cap \ker D_{A^{*k}} = \{0\} \iff \ldots \ker D_{A^{*N}} = \{0\}.
\end{aligned}$$

Theorem 4.2. [9]. *Let A be a completely non-unitary contraction in a separable Hilbert space $\mathfrak{H}$. Assume* $\ker D_A \neq \{0\}$ *and let the contractions $A_{n,m}$ be defined by* (4.1) *and* (4.2). *Then the characteristic functions of the operators*

$$A_{n,0}, A_{n-1,1}, \ldots, A_{n-m,m}, \ldots A_{1,n-1}, A_{0,n}$$

coincide with the pure part of the n-th Schur iterate of the characteristic function $\Phi(\lambda)$ of A. Moreover, each operator from the set $\{A_{n-k,k}\}_{k=0}^n$ is

(1) *a unilateral shift (respect., co-shift) if and only if the n-th Schur parameter Γ_n of Φ is isometric (respect., co-isometric),*
(2) *the orthogonal sum of a unilateral shift and co-shift if and only if*

$$\mathfrak{D}_{\Gamma_{n-1}} \neq \{0\},\ \mathfrak{D}_{\Gamma^*_{n-1}} \neq \{0\} \quad \textit{and} \quad \Gamma_m = 0 \quad \textit{for all} \quad m \geq n. \tag{4.4}$$

Each subspace from the set $\{\mathfrak{H}_{n-k,k}\}_{k=0}^n$ is trivial if and only if Γ_n is unitary.

Theorem 4.3. [9]. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and let*

$$\tau_0 = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

be a simple conservative realization of Θ. Then the Schur parameters $\{\Gamma_n\}_{n\geq 1}$ of Θ can be calculated as follows

$$\begin{aligned}
&\Gamma_1 = D^{-1}_{\Gamma^*_0} C \left(D^{-1}_{\Gamma_0} B^* \right)^*,\ \Gamma_2 = D^{-1}_{\Gamma^*_1} D^{-1}_{\Gamma^*_0} CA \left(D^{-1}_{\Gamma_1} D^{-1}_{\Gamma_0} \left(B^* {\upharpoonright}\, \mathfrak{H}_{1,0} \right) \right)^*, \ldots, \\
&\Gamma_n = D^{-1}_{\Gamma^*_{n-1}} \cdots D^{-1}_{\Gamma^*_0} CA^{n-1} \left(D^{-1}_{\Gamma_{n-1}} \cdots D^{-1}_{\Gamma_0} \left(B^* {\upharpoonright}\, \mathfrak{H}_{n-1,0} \right) \right)^*, \ldots .
\end{aligned} \tag{4.5}$$

Here the operators $D^{-1}_{\Gamma_k}$ and $D^{-1}_{\Gamma^*_k}, k=0,1,\ldots$ are the Moore-Penrose pseudo inverses, the operator

$$\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n-1,0}\right)\right)^*\in\mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{H}_{n-1,0})$$

is the adjoint to the operator

$$D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n-1,0}\right)\in\mathbf{L}(\mathfrak{H}_{n-1,0},\mathfrak{D}_{\Gamma_{n-1}}),$$

and

$$\begin{aligned}&\operatorname{ran}\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n,0}\right)\right)\subset\operatorname{ran}D_{\Gamma_n},\\&\operatorname{ran}\left(D^{-1}_{\Gamma^*_{n-1}}\cdots D^{-1}_{\Gamma^*_0}\left(C{\upharpoonright}\mathfrak{H}_{0,n}\right)\right)\subset\operatorname{ran}D_{\Gamma^*_n}\end{aligned}$$

for every $n\ge1$. Moreover, for each $n\ge1$ the unitarily equivalent simple conservative systems

(4.6)

$$\tau^{(k)}_n=\left\{\begin{bmatrix}\Gamma_n & D^{-1}_{\Gamma^*_{n-1}}\cdots D^{-1}_{\Gamma^*_0}(CA^{n-k})\\ A^k\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n,0}\right)\right)^* & A_{n-k,k}\end{bmatrix};\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{D}_{\Gamma^*_{n-1}},\mathfrak{H}_{n-k,k}\right\},$$

$k=0,1,\ldots,n$

are realizations of the n-th Schur iterate Θ_n of Θ. Here the operator

$$B_n=\left(D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n,0}\right)\right)^*\in\mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}},\mathfrak{H}_{n,0})$$

is the adjoint to the operator

$$D^{-1}_{\Gamma_{n-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{n,0}\right)\in\mathbf{L}(\mathfrak{H}_{n,0},\mathfrak{D}_{\Gamma_{n-1}}).$$

Note that if

1) Γ_m is isometric then $\mathfrak{D}_{\Gamma_n}=0$, $\Gamma^*_n=0\in\mathbf{L}(\mathfrak{D}_{\Gamma^*_m},\{0\})$, $\mathfrak{D}_{\Gamma^*_n}=\mathfrak{D}_{\Gamma^*_m}$, and $\mathfrak{H}_{0,n}=\mathfrak{H}_{0,m}$ for $n\ge m$. The unitarily equivalent observable conservative systems

$$\tau^{(k)}_m=\left\{\begin{bmatrix}\Gamma_m & D^{-1}_{\Gamma^*_{m-1}}\cdots D^{-1}_{\Gamma^*_0}(CA^{m-k})\\ 0 & A_{m-k,k}\end{bmatrix};\mathfrak{D}_{\Gamma_{m-1}},\mathfrak{D}_{\Gamma^*_{m-1}},\mathfrak{H}_{m-k,k}\right\},\ k=0,1,\ldots,m$$

have transfer functions $\Theta_m(\lambda)=\Gamma_m$ and the operators $A_{m-k,k}$ are unitarily equivalent co-shifts of multiplicity $\dim\mathfrak{D}_{\Gamma^*_m}$, the Schur iterates Θ_n are null operators from $\mathbf{L}(\{0\},\mathfrak{D}_{\Gamma^*_m})$ for $n\ge m+1$ and are transfer functions of the conservative observable system

$$\tau_{m+1}=\left\{\begin{bmatrix}0 & D^{-1}_{\Gamma^*_{m-1}}\cdots D^{-1}_{\Gamma^*_0}C\\ 0 & A_{0,m}\end{bmatrix};\{0\},\mathfrak{D}_{\Gamma^*_m},\mathfrak{H}_{0,m}\right\}.$$

2) Γ_m is co-isometric then $\mathfrak{D}_{\Gamma^*_n}=0$, $\mathfrak{D}_{\Gamma_n}=\mathfrak{D}_{\Gamma_m}$, and $\Gamma_n=0\in\mathbf{L}(\mathfrak{D}_{\Gamma_m},\{0\})$, $\mathfrak{H}_{n,0}=\mathfrak{H}_{m,0}$ for $n\ge m$. The unitarily equivalent controllable conservative systems

$$\tau^{(k)}_m=\left\{\begin{bmatrix}\Gamma_m & 0\\ A^k\left(D^{-1}_{\Gamma_{m-1}}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{m,0}\right)\right)^* & A_{m-k,k}\end{bmatrix};\mathfrak{D}_{\Gamma_{m-1}},\mathfrak{D}_{\Gamma^*_{m-1}},\mathfrak{H}_{m-k,k}\right\}$$

have transfer functions $\Theta_m(\lambda)=\Gamma_m$ and the operators $A_{m-k,k}$ are unitarily equivalent unilateral shifts of multiplicity $\dim\mathfrak{D}_{\Gamma_m}$, the Schur iterates Θ_n are null operators from $\mathbf{L}(\mathfrak{D}_{\Gamma_m},\{0\})$ for $n\ge m+1$ and are transfer functions of the conservative controllable system

$$\tau_{m+1}=\left\{\begin{bmatrix}0 & 0\\ \left(D^{-1}_{\Gamma_m}\cdots D^{-1}_{\Gamma_0}\left(B^*{\upharpoonright}\mathfrak{H}_{m+1,0}\right)\right)^* & A_{m,0}\end{bmatrix};\mathfrak{D}_{\Gamma_m},\{0\},\mathfrak{H}_{m,0}\right\}.$$

We also mention that if $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$, $\Gamma_0 = \Theta(0)$, $\Theta_1(\lambda)$ is the first Schur iterate of Θ, and if

$$\tau = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

is a simple conservative system with transfer function Θ, then the simple conservative systems

$$(4.7) \qquad \begin{array}{l} \zeta_{0,1} = \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0} C (D^{-1}_{\Gamma_0} B^*)^* & D^{-1}_{\Gamma^*_0} C \upharpoonright \ker D_{A^*} \\ A P_{\ker D_A} D^{-1}_{A^*} B & P_{\ker D_{A^*}} A \upharpoonright \ker D_{A^*} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_{A^*} \right\}, \\ \zeta_{1,0} = \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0} C (D^{-1}_{\Gamma_0} B^*)^* & D^{-1}_{\Gamma^*_0} C A \upharpoonright \ker D_A \\ P_{\ker D_A} D^{-1}_{A^*} B & P_{\ker D_A} A \upharpoonright \ker D_A \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_A \right\} \end{array}$$

have transfer functions $\Theta_1(\lambda)$ (see [9]). Here the operators $D^{-1}_{\Gamma_0}$, $D^{-1}_{\Gamma^*_0}$, and $D^{-1}_{A^*}$ are the Moore–Penrose pseudo-inverses. In the sequel the transformations of the conservative system

$$\tau \longrightarrow \zeta_{0,1}, \ \tau \longrightarrow \zeta_{1,0}$$

we will denote by $\Omega_{0,1}(\tau)$ and $\Omega_{1,0}(\tau)$, respectively.

Remark 4.4. The problem of isometric, co-isometric, and conservative realizations of the Schur iterates for a scalar function from the generalized Schur class has been studied in [2], [3], [4], [5]. For a scalar finite Blaschke product the realizations of the Schur iterates are constructed in [43].

5. Block operator CMV matrices and conservative realizations of the Schur class function (the case when the operator Γ_n is neither an isometry nor a co-isometry for each n)

Let

$$\Gamma_0 \in \mathbf{L}(\mathfrak{M}, \mathfrak{N}), \ \Gamma_n \in \mathbf{L}(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma^*_{n-1}}), \ n \geq 1$$

be a choice sequence. In this and next Section 6 we are going to construct by means of $\{\Gamma_n\}_{n\geq 0}$ two unitary equivalent simple conservative systems with such transfer function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ that $\{\Gamma_n\}_{n\geq 0}$ are its Schur parameters. In particular, this leads to the existence part of Theorem 1.2 and to the well known result that any $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ admits a realization as the transfer function of a simple conservative system. We begin with constructions of block operator CMV matrices for given choice sequence $\{\Gamma_n\}_{n\geq 0}$ and will suppose that all operators Γ_n are neither isometries nor co-isometries. We will use the well known constructions of finite and infinite orthogonal sums of Hilbert spaces. Namely, if $\{H_k\}_{k=1}^{\infty}$ is a given sequence of Hilbert spaces, then

$$\mathfrak{H} = \sum_{k=1}^{N} \bigoplus H_k$$

is the Hilbert space with the inner product

$$(f, g) = \sum_{k=0}^{N} (f_k, g_k)_{H_k}$$

for $f = (f_1, \ldots, f_N)^T$ and $g = (g_1, \ldots, g_N)^T$, $f_k, g_k \in H_k$, $k = 1, \ldots, N$ and the norm

$$||f||^2 = \sum_{k=0}^{N} ||f_k||^2_{H_k}.$$

The Hilbert space

$$\mathfrak{H}=\sum_{k=0}^{\infty}\bigoplus H_k$$

consists of all vectors of the form $f=(f_1,f_2,\ldots)^T$, $f_k\in H_k$, $k=1,2,\ldots$, such that

$$||f||^2=\sum_{k=1}^{\infty}||f_k||^2_{H_k}<\infty.$$

The inner product is given by

$$(f,g)=\sum_{k=1}^{\infty}(f_k,g_k)_{H_k}.$$

5.1. **Block operator CMV matrices.** Define the Hilbert spaces

$$\mathfrak{H}_0=\mathfrak{H}_0(\{\Gamma_n\}_{n\geq 0}):=\sum_{n\geq 0}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{matrix},\ \widetilde{\mathfrak{H}}_0=\widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\geq 0}):=\sum_{n\geq 0}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{matrix}. \tag{5.1}$$

From these definitions it follows, that

$$\widetilde{\mathfrak{H}}_0(\{\Gamma^*_n\}_{n\geq 0})=\mathfrak{H}_0(\{\Gamma_n\}_{n\geq 0}),\ \mathfrak{H}_0(\{\Gamma^*_n\}_{n\geq 0})=\widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\geq 0}).$$

The spaces $\mathfrak{N}\bigoplus\mathfrak{H}_0$ and $\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0$ we represent in the form

$$\mathfrak{N}\bigoplus\mathfrak{H}_0=\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_{\Gamma_0}\end{matrix}\bigoplus\sum_{n\geq 1}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma^*_{2n-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n}}\end{matrix},$$

$$\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0=\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_0}\end{matrix}\bigoplus\sum_{n\geq 1}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma_{2n-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n}}\end{matrix}.$$

Let

$$\mathbf{J}_{\Gamma_0}=\begin{bmatrix}\Gamma_0 & D_{\Gamma^*_0}\\ D_{\Gamma_0} & -\Gamma^*_0\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_0}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{D}_{\Gamma_0}\end{matrix},$$

$$\mathbf{J}_{\Gamma_k}=\begin{bmatrix}\Gamma_k & D_{\Gamma^*_k}\\ D_{\Gamma_k} & -\Gamma^*_k\end{bmatrix}:\begin{matrix}\mathfrak{D}_{\Gamma_{k-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_k}\end{matrix}\to\begin{matrix}\mathfrak{D}_{\Gamma^*_{k-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_k}\end{matrix},\ k=1,2,\ldots$$

be the elementary rotations. Define the following unitary operators

$$\begin{aligned}&\mathcal{M}_0=\mathcal{M}_0(\{\Gamma_n\}_{n\geq 0}):=I_{\mathfrak{M}}\bigoplus\sum_{n\geq 1}\bigoplus\mathbf{J}_{\Gamma_{2n-1}}:\mathfrak{M}\bigoplus\mathfrak{H}_0\to\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0,\\ &\widetilde{\mathcal{M}}_0=\widetilde{\mathcal{M}}_0(\{\Gamma_n\}_{n\geq 0}):=I_{\mathfrak{N}}\bigoplus\sum_{n\geq 1}\bigoplus\mathbf{J}_{\Gamma_{2n-1}}:\mathfrak{N}\bigoplus\mathfrak{H}_0\to\mathfrak{N}\bigoplus\widetilde{\mathfrak{H}}_0,\\ &\mathcal{L}_0=\mathcal{L}_0(\{\Gamma_n\}_{n\geq 0}):=\mathbf{J}_{\Gamma_0}\bigoplus\sum_{n\geq 1}\bigoplus\mathbf{J}_{\Gamma_{2n}}:\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0\to\mathfrak{N}\bigoplus\mathfrak{H}_0.\end{aligned} \tag{5.2}$$

Observe that

$$(\mathcal{L}_0(\{\Gamma_n\}_{n\geq 0}))^*=\mathcal{L}_0(\{\Gamma^*_n\}_{n\geq 0}).$$

Let

$$\mathcal{V}_0=\mathcal{V}_0(\{\Gamma_n\}_{n\ge 0}):=\sum_{n\ge 1}\bigoplus \mathbf{J}_{\Gamma_{2n-1}}:\mathfrak{H}_0\to\widetilde{\mathfrak{H}}_0. \tag{5.3}$$

Clearly, the operator $\mathcal{V}_0$ is unitary and

$$\mathcal{M}_0=I_{\mathfrak{M}}\bigoplus\mathcal{V}_0,\ \widetilde{\mathcal{M}}_0=I_{\mathfrak{N}}\bigoplus\mathcal{V}_0. \tag{5.4}$$

It follows that

$$\left(\widetilde{\mathcal{M}}_0(\{\Gamma_n\}_{n\ge 0})\right)^*=\mathcal{M}_0(\{\Gamma^*_n\}_{n\ge 0}),\ (\mathcal{M}_0(\{\Gamma_n\}_{n\ge 0}))^*=\widetilde{\mathcal{M}}_0(\{\Gamma^*_n\}_{n\ge 0})$$

Finally define the unitary operators

$$\begin{array}{l}\mathcal{U}_0=\mathcal{U}_0(\{\Gamma_n\}_{n\ge 0}):=\mathcal{L}_0\mathcal{M}_0:\mathfrak{M}\bigoplus\mathfrak{H}_0\to\mathfrak{N}\bigoplus\mathfrak{H}_0,\\ \widetilde{\mathcal{U}}_0=\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 0}):=\widetilde{\mathcal{M}}_0\mathcal{L}_0:\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0\to\mathfrak{N}\bigoplus\widetilde{\mathfrak{H}}_0.\end{array} \tag{5.5}$$

By calculations we get

$$\mathcal{U}_0=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots\\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots\\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 & D_{\Gamma_2^*}D_{\Gamma_3^*} & 0 & 0 & 0 & \ldots & \ldots\\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 & -\Gamma_2^*D_{\Gamma_3^*} & 0 & 0 & 0 & \ldots & \ldots\\ 0 & 0 & 0 & \Gamma_4 D_{\Gamma_3} & -\Gamma_4\Gamma_3^* & D_{\Gamma_4^*}\Gamma_5 & D_{\Gamma_4^*}D_{\Gamma_5^*} & 0 & \ldots & \ldots\\ 0 & 0 & 0 & D_{\Gamma_4}D_{\Gamma_3} & -D_{\Gamma_4}\Gamma_3^* & -\Gamma_4^*\Gamma_5 & -\Gamma_4 D_{\Gamma_5^*} & 0 & \ldots & \ldots\\ 0 & 0 & 0 & 0 & 0 & \Gamma_6 D_{\Gamma_5} & -\Gamma_6\Gamma_5^* & D_{\Gamma_6^*}\Gamma_7 & D_{\Gamma_6^*}D_{\Gamma_7^*} & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}$$

and

$$\widetilde{\mathcal{U}}_0=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots\\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & \ldots & \ldots\\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & -\Gamma_1^*D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & \ldots & \ldots\\ 0 & 0 & \Gamma_3 D_{\Gamma_2} & -\Gamma_3\Gamma_2^* & D_{\Gamma_3^*}\Gamma_4 & D_{\Gamma_3^*}D_{\Gamma_4^*} & 0 & 0 & \ldots & \ldots\\ 0 & 0 & D_{\Gamma_3}D_{\Gamma_2} & -D_{\Gamma_3}\Gamma_2^* & -\Gamma_3^*\Gamma_4 & -\Gamma_3^*D_{\Gamma_4^*} & 0 & 0 & \ldots & \ldots\\ 0 & 0 & 0 & 0 & \Gamma_5 D_{\Gamma_4} & -\Gamma_5\Gamma_4^* & D_{\Gamma_5^*}\Gamma_6 & D_{\Gamma_5^*}D_{\Gamma_6^*} & 0 & \ldots\\ 0 & 0 & 0 & 0 & D_{\Gamma_5}D_{\Gamma_4} & -D_{\Gamma_5}\Gamma_4^* & -\Gamma_5^*\Gamma_6 & -\Gamma_5^*D_{\Gamma_6^*} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

Let

$$\mathcal{C}_0=\begin{bmatrix}D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*}\end{bmatrix}:\begin{array}{c}\mathfrak{D}_{\Gamma_0}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array}\to\mathfrak{N},\ \mathcal{A}_0=\begin{bmatrix}D_{\Gamma_0}\\ 0\end{bmatrix}:\mathfrak{M}\to\begin{array}{c}\mathfrak{D}_{\Gamma_0}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array},$$

$$
\left\{
\begin{aligned}
\mathcal{B}_n &= \begin{bmatrix} -\Gamma^*_{2n-2}\Gamma_{2n-1} & -\Gamma^*_{2n-2}D_{\Gamma^*_{2n-1}} \\ \Gamma_{2n}D_{\Gamma_{2n-1}} & -\Gamma_{2n}\Gamma^*_{2n-1} \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n-1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n-1}} \end{array}, \\
\mathcal{C}_n &= \begin{bmatrix} 0 & 0 \\ D_{\Gamma^*_{2n}}\Gamma_{2n+1} & D_{\Gamma^*_{2n}}D_{\Gamma^*_{2n+1}} \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n+1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n-1}} \end{array}, \\
\mathcal{A}_n &= \begin{bmatrix} D_{\Gamma_{2n}}D_{\Gamma_{2n-1}} & -D_{\Gamma_{2n}}\Gamma^*_{2n-1} \\ 0 & 0 \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n-1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n+1}} \end{array},
\end{aligned}
\right. \tag{5.6}
$$

$$
\widetilde{\mathcal{C}}_0 = \begin{bmatrix} D_{\Gamma^*_0} & 0 \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma^*_0} \\ \oplus \\ \mathfrak{D}_{\Gamma_1} \end{array} \to \mathfrak{N},\ \widetilde{\mathcal{A}}_0 = \begin{bmatrix} \Gamma_1 D_{\Gamma_0} \\ D_{\Gamma_1}D_{\Gamma_0} \end{bmatrix} : \mathfrak{M} \to \begin{array}{c} \mathfrak{D}_{\Gamma^*_0} \\ \oplus \\ \mathfrak{D}_{\Gamma_1} \end{array},
$$

$$
\left\{
\begin{aligned}
\widetilde{\mathcal{B}}_n &= \begin{bmatrix} -\Gamma_{2n-1}\Gamma^*_{2n-2} & D_{\Gamma^*_{2n-1}}\Gamma_{2n} \\ -D_{\Gamma_{2n-1}}\Gamma^*_{2n-2} & -\Gamma^*_{2n-1}\Gamma_{2n} \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n-1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n-1}} \end{array}, \\
\widetilde{\mathcal{C}}_n &= \begin{bmatrix} D_{\Gamma^*_{2n-1}}D_{\Gamma^*_{2n}} & 0 \\ -\Gamma^*_{2n-1}D_{\Gamma^*_{2n}} & 0 \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n+1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n-1}} \end{array}, \\
\widetilde{\mathcal{A}}_n &= \begin{bmatrix} 0 & \Gamma_{2n+1}D_{\Gamma_{2n}} \\ 0 & D_{\Gamma_{2n+1}}D_{\Gamma_{2n}} \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n-2}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n-1}} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma^*_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n+1}} \end{array}
\end{aligned}
\right. \tag{5.7}
$$

It is easy to see that the operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ take the following three-diagonal block operator matrix form

$$
\mathcal{U}_0 = \mathcal{U}_0\left(\{\Gamma_n\}_{n\ge 0}\right) = \begin{bmatrix} \Gamma_0 & \mathcal{C}_0 & 0 & 0 & 0 & \cdot & \cdot \\ \mathcal{A}_0 & \mathcal{B}_1 & \mathcal{C}_1 & 0 & 0 & \cdot & \cdot \\ 0 & \mathcal{A}_1 & \mathcal{B}_2 & \mathcal{C}_2 & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},
$$

$$
\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0\left(\{\Gamma_n\}_{n\ge 0}\right) = \begin{bmatrix} \Gamma_0 & \widetilde{\mathcal{C}}_0 & 0 & 0 & 0 & \cdot & \cdot \\ \widetilde{\mathcal{A}}_0 & \widetilde{\mathcal{B}}_1 & \widetilde{\mathcal{C}}_1 & 0 & 0 & \cdot & \cdot \\ 0 & \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{B}}_2 & \widetilde{\mathcal{C}}_2 & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.
$$

The block operator matrices $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ we will call *block operator CMV matrices*. Observe that

$$
\widetilde{M}_0\mathcal{U}_0 = \widetilde{\mathcal{U}}_0\mathcal{M}_0, \tag{5.8}
$$

and the following equalities hold true

$$
\left(\mathcal{U}_0(\{\Gamma_n\}_{n\ge 0})\right)^* = \widetilde{\mathcal{U}}_0(\{\Gamma^*_n\}_{n\ge 0}),\ \left(\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 0})\right)^* = \mathcal{U}_0(\{\Gamma^*_n\}_{n\ge 0}) \tag{5.9}
$$

Therefore the matrix $\widetilde{\mathcal{U}}_0$ can be obtained from $\mathcal{U}_0$ by passing to the adjoint $\mathcal{U}_0^*$ and then by replacing Γ_n (respect., Γ^*_n) by Γ^*_n (respect., Γ_n) for all n. In the case when the choice sequence consists of complex numbers from the unit disk the matrix $\widetilde{\mathcal{U}}_0$ is the transpose to $\mathcal{U}_0$, i.e., $\widetilde{\mathcal{U}}_0 = \mathcal{U}_0^t$.

5.2. **Truncated block operator CMV matrices.** Define two contractions

$$\mathcal{T}_0=\mathcal{T}_0(\{\Gamma_n\}_{n\ge0}):=P_{\mathfrak{H}_0}\mathcal{U}_0{\upharpoonright}\,\mathfrak{H}_0:\mathfrak{H}_0\to\mathfrak{H}_0, \tag{5.10}$$

$$\widetilde{\mathcal{T}}_0=\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\ge0}):=P_{\widetilde{\mathfrak{H}}_0}\widetilde{\mathcal{U}}_0{\upharpoonright}\,\widetilde{\mathfrak{H}}_0:\widetilde{\mathfrak{H}}_0\to\widetilde{\mathfrak{H}}_0. \tag{5.11}$$

The operators $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$ take on the three-diagonal block operator matrix forms

$$\mathcal{T}_0=\begin{bmatrix}\mathcal{B}_1&\mathcal{C}_1&0&0&0&\cdot\\ \mathcal{A}_1&\mathcal{B}_2&\mathcal{C}_2&0&0&\cdot\\ 0&\mathcal{A}_2&\mathcal{B}_3&\mathcal{C}_3&0&\cdot\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\end{bmatrix},\quad \widetilde{\mathcal{T}}_0=\begin{bmatrix}\widetilde{\mathcal{B}}_1&\widetilde{\mathcal{C}}_1&0&0&0&\cdot\\ \widetilde{\mathcal{A}}_1&\widetilde{\mathcal{B}}_2&\widetilde{\mathcal{C}}_2&0&0&\cdot\\ 0&\widetilde{\mathcal{A}}_2&\widetilde{\mathcal{B}}_3&\widetilde{\mathcal{C}}_3&0&\cdot\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\end{bmatrix},$$

where $\mathcal{A}_n,\mathcal{B}_n,\mathcal{C}_n$, $\widetilde{\mathcal{A}}_n,\widetilde{\mathcal{B}}_n$, and $\widetilde{\mathcal{C}}_n$ are given by (5.6) and (5.7). Since the matrices $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$ are obtained from $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ by deleting the first rows and the first columns, we will call them *truncated block operator CMV matrices*. Observe that from the definitions of $\mathcal{L}_0$, $\mathcal{M}_0$, $\widetilde{\mathcal{M}}_0$, $\mathcal{T}_0$, and $\widetilde{\mathcal{T}}_0$ it follows that $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$ are products of two block-diagonal matrices

$$\mathcal{T}_0=\mathcal{T}_0(\{\Gamma_n\}_{n\ge0})=\begin{bmatrix}-\Gamma_0^*&&&&&\\ &\mathbf{J}_{\Gamma_2}&&&&\\ &&\mathbf{J}_{\Gamma_4}&&&\\ &&&\ddots&&\\ &&&&\mathbf{J}_{\Gamma_{2n}}&\\ &&&&&\ddots\end{bmatrix}\begin{bmatrix}\mathbf{J}_{\Gamma_1}&&&\\ &\mathbf{J}_{\Gamma_3}&&\\ &&\ddots&\\ &&&\mathbf{J}_{\Gamma_{2n+1}}\\ &&&&\ddots\end{bmatrix}, \tag{5.12}$$

$$\widetilde{\mathcal{T}}_0=\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\ge0})=\begin{bmatrix}\mathbf{J}_{\Gamma_1}&&&\\ &\mathbf{J}_{\Gamma_3}&&\\ &&\ddots&\\ &&&\mathbf{J}_{\Gamma_{2n+1}}\\ &&&&\ddots\end{bmatrix}\begin{bmatrix}-\Gamma_0^*&&&&&\\ &\mathbf{J}_{\Gamma_2}&&&&\\ &&\mathbf{J}_{\Gamma_4}&&&\\ &&&\ddots&&\\ &&&&\mathbf{J}_{\Gamma_{2n}}&\\ &&&&&\ddots\end{bmatrix}. \tag{5.13}$$

In particular, it follows that

$$(\mathcal{T}_0(\{\Gamma_n\}_{n\ge0}))^*=\widetilde{\mathcal{T}}_0(\{\Gamma_n^*\}_{n\ge0}). \tag{5.14}$$

From (5.12) and (5.13) we have

$$\mathcal{V}_0\mathcal{T}_0=\widetilde{\mathcal{T}}_0\mathcal{V}_0,$$

where the unitary operator $\mathcal{V}_0$ is defined by (5.3). Therefore, the operators $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$ are unitarily equivalent.

Proposition 5.1. *Let $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ and let $\{\Gamma_n\}_{n\ge0}$ be the Schur parameters of Θ. Suppose Γ_n is neither isometric nor co-isometric for each n. Let the function $\Omega\in\mathbf{S}(\mathfrak{K},\mathfrak{L})$ coincides with Θ and Let $\{G_n\}_{n\ge0}$ be the Schur parameters of Ω. Then truncated block operator CMV matrices $\mathcal{T}_0(\{\Gamma_n\}_{n\ge0})$ and $\mathcal{T}_0(\{G_n\}_{n\ge0})$ (respect., $\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\ge0})$ and $\widetilde{\mathcal{T}}_0(\{G_n\}_{n\ge0})$) are unitarily equivalent.*

Proof. Since $\Omega(\lambda) = V\Theta(\lambda)U$, where $U \in \mathbf{L}(\mathfrak{K}, \mathfrak{M})$ and $V \in \mathbf{L}(\mathfrak{N}, \mathfrak{L})$ are unitary operators, we get relations (2.2). It follows that $\mathfrak{D}_{G_n} \neq \{0\}$ and $\mathfrak{D}_{G_n^*} \neq \{0\}$ for all n. Hence, we have

$$\mathbf{J}_{G_n}\begin{bmatrix} U^* & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U^* \end{bmatrix}\mathbf{J}_{\Gamma_n},\ n = 0, 1, \ldots. \tag{5.15}$$

Define the Hilbert space

$$\mathfrak{H}_0^\Omega = \mathfrak{H}_0(\{G_n\}_{n\geq 0}) := \sum_{n\geq 0}\bigoplus \begin{matrix} \mathfrak{D}_{G_{2n}} \\ \oplus \\ \mathfrak{D}_{G^*_{2n+1}} \end{matrix}$$

and truncated block operator CMV matrix

$$\mathcal{T}_0(\{G_n\}_{n\geq 0}) := \begin{bmatrix} -G_0^* & & & & & \\ & \mathbf{J}_{G_2} & & & & \\ & & \mathbf{J}_{G_4} & & & \\ & & & \ddots & & \\ & & & & \mathbf{J}_{G_{2n}} & \\ & & & & & \ddots \end{bmatrix}\begin{bmatrix} \mathbf{J}_{G_1} & & & \\ & \mathbf{J}_{G_3} & & \\ & & \ddots & \\ & & & \mathbf{J}_{G_{2n+1}} \\ & & & \ddots \end{bmatrix},$$

Define the unitary operator

$$\mathcal{W} = \begin{bmatrix} U^* & & & & \\ & V & & & \\ & & U^* & & \\ & & & V & \\ & & & & \ddots \end{bmatrix} : \mathfrak{H}_0 \to \mathfrak{H}_0^\Omega$$

From (5.12) and (5.15) we obtain

$$\mathcal{W}\mathcal{T}_0(\{\Gamma_n\}_{n\geq 0}) = \mathcal{T}(\{G_n\}_{n\geq 0})\mathcal{W}.$$

Thus $\mathcal{T}_0(\{\Gamma_n\}_{n\geq 0})$ and $\mathcal{T}(\{G_n\}_{n\geq 0})$ are unitarily equivalent. □

Now we are going to find the defect operators and defect subspaces for $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$. Let $\mathbf{f} = (\vec{f}_0, \vec{f}_1, \ldots)^T \in \mathfrak{H}_0$, where

$$\vec{f}_n = \begin{bmatrix} h_n \\ g_n \end{bmatrix} \in \begin{matrix} \mathfrak{D}_{\Gamma_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n+1}} \end{matrix},\ n = 0, 1, \ldots.$$

Then

$$\begin{aligned} &||\mathbf{f}||^2 - ||\mathcal{T}_0\mathbf{f}||^2 = ||P_{\mathfrak{N}}\mathcal{U}_0\mathbf{f}||^2 = \left\|\mathcal{C}_0\begin{bmatrix} h_0 \\ g_0 \end{bmatrix}\right\|^2 = ||D_{\Gamma_0^*}(\Gamma_1 h_0 + D_{\Gamma_1^*}g_0)||^2, \\ &||\mathbf{f}||^2 - ||\mathcal{T}_0^*\mathbf{f}||^2 = ||P_{\mathfrak{M}}\mathcal{U}_0^*\mathbf{f}||^2 = \left\|\mathcal{A}_0^*\begin{bmatrix} h_0 \\ g_0 \end{bmatrix}\right\|^2 = ||D_{\Gamma_0}h_0||^2. \end{aligned} \tag{5.16}$$

Let $\mathbf{x} = (x_0, x_1, \ldots)^T \in \widetilde{\mathfrak{H}}_0$, where

$$x_n = \begin{bmatrix} h_n \\ g_n \end{bmatrix} \in \begin{matrix} \mathfrak{D}_{\Gamma^*_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma^*_{2n+1}} \end{matrix},\ n = 0, 1, \ldots.$$

Then

$$
\begin{aligned}
&||\mathbf{x}||^2-||\widetilde{\mathcal{T}}_0\mathbf{x}||^2=||P_{\mathfrak{N}}\widetilde{\mathcal{U}}_0\mathbf{x}||^2=\left\|\widetilde{\mathcal{C}}_0\begin{bmatrix}h_0\\ g_0\end{bmatrix}\right\|^2=||D_{\Gamma_0^*}h_0||^2,\\
&||\mathbf{x}||^2-||\widetilde{\mathcal{T}}_0^*\mathbf{x}||^2=||P_{\mathfrak{M}}\widetilde{\mathcal{U}}_0^*\mathbf{x}||^2=\left\|\widetilde{\mathcal{A}}_0^*\begin{bmatrix}h_0\\ g_0\end{bmatrix}\right\|^2=||D_{\Gamma_0}(\Gamma_1^*h_0+D_{\Gamma_1}g_0)||^2.
\end{aligned}
$$

Now from Proposition 2.1 it follows that

$$
\begin{cases}
\ker D_{\mathcal{T}_0}=\left\{\begin{bmatrix}D_{\Gamma_1}\varphi\\ -\Gamma_1\varphi\end{bmatrix},\ \varphi\in\mathfrak{D}_{\Gamma_1}\right\}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array},\\
\ker D_{\mathcal{T}_0^*}=\mathfrak{D}_{\Gamma_1^*}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array},\\
\mathfrak{D}_{\mathcal{T}_0}=\left\{\begin{bmatrix}\Gamma_1^*\psi\\ D_{\Gamma_1^*}\psi\end{bmatrix},\ \psi\in\mathfrak{D}_{\Gamma_0^*}\right\}\bigoplus\vec{0},\ \vec{0}\in\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array},\\
\mathfrak{D}_{\mathcal{T}_0^*}=\mathfrak{D}_{\Gamma_0}\bigoplus\vec{0},\ \vec{0}\in\mathfrak{D}_{\Gamma_1^*}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array}.
\end{cases}
\tag{5.17}
$$

$$
\begin{cases}
\ker D_{\widetilde{\mathcal{T}}_0}=\mathfrak{D}_{\Gamma_1}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{array},\\
\ker D_{\widetilde{\mathcal{T}}_0^*}=\left\{\begin{bmatrix}D_{\Gamma_1^*}\varphi\\ -\Gamma_1^*\varphi\end{bmatrix},\ \varphi\in\mathfrak{D}_{\Gamma_0^*}\right\}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{array},\\
\mathfrak{D}_{\widetilde{\mathcal{T}}_0}=\mathfrak{D}_{\Gamma_0^*}\bigoplus\vec{0},\ \vec{0}\in\mathfrak{D}_{\Gamma_1}\bigoplus\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{array},\\
\mathfrak{D}_{\widetilde{\mathcal{T}}_0^*}=\left\{\begin{bmatrix}\Gamma_1\psi\\ D_{\Gamma_1}\psi\end{bmatrix},\ \psi\in\mathfrak{D}_{\Gamma_0}\right\}\bigoplus\vec{0},\ \vec{0}\in\sum\limits_{n\ge 1}\bigoplus\begin{array}{c}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{array}.
\end{cases}
\tag{5.18}
$$

5.3. **Simple conservative realizations of the Schur class function by means of its Schur parameters.** Let

$$
\begin{aligned}
&\mathcal{G}_0=\mathcal{G}_0(\{\Gamma_n\}_{n\ge}0)=\begin{bmatrix}D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & \ldots\end{bmatrix}:\mathfrak{H}_0\to\mathfrak{N},\\
&\widetilde{\mathcal{G}}_0=\widetilde{\mathcal{G}}_0(\{\Gamma_n\}_{n\ge 0})=\begin{bmatrix}D_{\Gamma_0^*} & 0 & 0 & \ldots\end{bmatrix}:\widetilde{\mathfrak{H}}_0\to\mathfrak{N},\\
&\mathcal{F}_0=\mathcal{F}_0(\{\Gamma_n\}_{n\ge 0})=\begin{bmatrix}D_{\Gamma_0}\\ 0\\ 0\\ \vdots\end{bmatrix}:\mathfrak{M}\to\mathfrak{H}_0,\ \widetilde{\mathcal{F}}_0=\widetilde{\mathcal{F}}_0(\{\Gamma_n\}_{n\ge 0})=\begin{bmatrix}\Gamma_1D_{\Gamma_0}\\ D_{\Gamma_1}D_{\Gamma_0}\\ 0\\ 0\\ \vdots\end{bmatrix}:\mathfrak{M}\to\widetilde{\mathfrak{H}}_0.
\end{aligned}
$$

The operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ can be represented by 2×2 block operator matrices

$$\mathcal{U}_0 = \begin{bmatrix}\Gamma_0 & \mathcal{G}_0\\ \mathcal{F}_0 & \mathcal{T}_0\end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{H}_0\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus\\ \mathfrak{H}_0\end{array},$$
$$\widetilde{\mathcal{U}}_0 = \begin{bmatrix}\Gamma_0 & \widetilde{\mathcal{G}}_0\\ \widetilde{\mathcal{F}}_0 & \widetilde{\mathcal{T}}_0\end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus\\ \widetilde{\mathfrak{H}}_0\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus\\ \widetilde{\mathfrak{H}}_0\end{array}.$$

Define the following conservative systems

$$\begin{aligned}\zeta_0 &= \left\{\begin{bmatrix}\Gamma_0 & \mathcal{G}_0\\ \mathcal{F}_0 & \mathcal{T}_0\end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_0\right\} = \left\{\mathcal{U}_0(\{\Gamma_n\}_{n\ge 0}); \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_0(\{\Gamma_n\}_{n\ge 0})\right\},\\ \widetilde{\zeta}_0 &= \left\{\begin{bmatrix}\Gamma_0 & \widetilde{\mathcal{G}}_0\\ \widetilde{\mathcal{F}}_0 & \widetilde{\mathcal{T}}_0\end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \widetilde{\mathfrak{H}}_0\right\} = \left\{\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 0}); \mathfrak{M}, \mathfrak{N}, \widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\ge 0})\right\}.\end{aligned} \tag{5.19}$$

The equalities (5.4) and (5.8) yield that systems ζ_0 and $\widetilde{\zeta}_0$ are unitarily equivalent. Hence, ζ_0 and $\widetilde{\zeta}_0$ have equal transfer functions.

Observe that

$$\mathcal{F}_0 = \begin{bmatrix} I_{\mathfrak{M}}\\ 0\\ 0\\ \vdots\end{bmatrix} D_{\Gamma_0},\ \mathcal{G}_0 = D_{\Gamma_0^*}\begin{bmatrix}\Gamma_1 & D_{\Gamma_1^*} & 0 & 0 & \ldots\end{bmatrix},$$

$$\widetilde{\mathcal{F}}_0 = \begin{bmatrix}\Gamma_1\\ D_{\Gamma_1}\\ 0\\ 0\\ \vdots\end{bmatrix} D_{\Gamma_0},\ \widetilde{\mathcal{G}}_0 = D_{\Gamma_0^*},\begin{bmatrix} I_{\mathfrak{N}} & 0 & 0 & 0 & \ldots\end{bmatrix}$$

and

$$\begin{bmatrix}\Gamma_1 & D_{\Gamma_1^*} & 0 & 0 & \ldots\end{bmatrix}\begin{bmatrix} I_{\mathfrak{M}}\\ 0\\ 0\\ \vdots\end{bmatrix} = \begin{bmatrix} I_{\mathfrak{N}} & 0 & 0 & 0 & \ldots\end{bmatrix}\begin{bmatrix}\Gamma_1\\ D_{\Gamma_1}\\ 0\\ 0\\ \vdots\end{bmatrix} = \Gamma_1.$$

Theorem 5.2. *The unitarily equivalent conservative systems ζ_0 and $\widetilde{\zeta}_0$ given by* (5.19) *are simple and the Schur parameters of the transfer function of ζ_0 and $\widetilde{\zeta}_0$ are $\{\Gamma_n\}_{n\ge 0}$.*

Proof. The main step is a proof that the systems $\Omega_{0,1}(\zeta_0)$ and $\Omega_{1,0}(\widetilde{\zeta}_0)$ given by (4.7) take the form

$$\begin{aligned}\Omega_{0,1}(\zeta_0) &= \left\{\widetilde{U}_0\left(\{\Gamma_n\}_{n\ge 1}\right), \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}, \widetilde{\mathfrak{H}}_0\left(\{\Gamma_n\}_{n\ge 1}\right)\right\},\\ \Omega_{1,0}(\widetilde{\zeta}_0) &= \left\{U_0\left(\{\Gamma_n\}_{n\ge 1}\right), \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}, \mathfrak{H}_0\left(\{\Gamma_n\}_{n\ge 1}\right)\right\}.\end{aligned} \tag{5.20}$$

First of all we will prove that the systems ζ_0 and $\widetilde{\zeta}_0$ are simple.

Define the subspaces

$$\mathfrak{H}_{2k-1}=\sum_{n\ge k}\bigoplus \begin{array}{l}\mathfrak{D}_{\Gamma^*_{2n-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n}}\end{array},\ \mathfrak{H}_{2k}=\sum_{n\ge k}\bigoplus \begin{array}{l}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{array},\ k=1,2,\ldots$$

Clearly, $\mathfrak{H}_0\supset\mathfrak{H}_1\supset\mathfrak{H}_2\supset\cdots\supset\mathfrak{H}_m\supset\cdots$. From (5.1) it follows the equality

$$\bigcap_{m\ge 0}\mathfrak{H}_m=\{0\}.$$

Let $\Gamma_{-1}=0:\mathfrak{M}\to\mathfrak{N}$. Then $\mathfrak{D}_{\Gamma_{-1}}=\mathfrak{M}$, $\mathfrak{D}_{\Gamma^*_{-1}}=\mathfrak{N}$. We can consider $\mathcal{U}_0$ as acting from $\mathfrak{D}_{\Gamma_{-1}}\oplus\mathfrak{H}_0$ onto $\mathfrak{D}_{\Gamma^*_{-1}}\oplus\mathfrak{H}_0$ and $\widetilde{\mathcal{U}}_0$ as acting from $\mathfrak{D}_{\Gamma_{-1}}\oplus\widetilde{\mathfrak{H}}_0$ onto $\mathfrak{D}_{\Gamma^*_{-1}}\oplus\widetilde{\mathfrak{H}}_0$. Fix $m\in\mathbb{N}$ and define

$$\Gamma^{(m)}_n=\Gamma_{n+m},\ n=-1,0,1,\ldots$$

Then $\{\Gamma^{(m)}_n\}_{n\ge 0}=\{\Gamma_k\}_{k\ge m}$, and

$$\begin{aligned}\mathfrak{H}_{2k-1}&=\widetilde{\mathfrak{H}}_0(\{\Gamma^{(2k-1)}_n\}_{n\ge 0})=\widetilde{\mathfrak{H}}_0\left(\{\Gamma_n\}_{n\ge 2k-1}\right),\\ \mathfrak{H}_{2k}&=\mathfrak{H}_0(\{\Gamma^{(2k)}_n\}_{n\ge 0})=\mathfrak{H}_0\left(\{\Gamma_n\}_{n\ge 2k}\right).\end{aligned}$$

Let

$$\mathcal{W}_{2k-1}=\widetilde{\mathcal{U}}_0(\{\Gamma^{(2k-1)}_n\}_{n\ge 0})=\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge 2k-1}):\ \begin{array}{l}\mathfrak{D}_{\Gamma_{2k-2}}\\ \oplus\\ \mathfrak{H}_{2k-1}\end{array}\to\begin{array}{l}\mathfrak{D}_{\Gamma^*_{2k-2}}\\ \oplus\\ \mathfrak{H}_{2k-1}\end{array},$$

$$\mathcal{W}_{2k}=\mathcal{U}_0(\{\Gamma^{(2k)}_n\}_{n\ge 0})=\mathcal{U}_0(\{\Gamma_n\}_{n\ge 2k}):\ \begin{array}{l}\mathfrak{D}_{\Gamma_{2k-1}}\\ \oplus\\ \mathfrak{H}_{2k}\end{array}\to\begin{array}{l}\mathfrak{D}_{\Gamma^*_{2k-1}}\\ \oplus\\ \mathfrak{H}_{2k}\end{array},\ k\ge 1.$$

Define the operators

$$\mathcal{T}_m=P_{\mathfrak{H}_m}\mathcal{W}_m{\upharpoonright}\mathfrak{H}_m,\ m=1,2,\ldots. \tag{5.21}$$

Then

$$\begin{aligned}\mathcal{T}_{2k-1}&=\widetilde{\mathcal{T}}_0(\{\Gamma^{(2k-1)}_n\}_{n\ge 0})=\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\ge 2k-1}),\\ \mathcal{T}_{2k}&=\mathcal{T}_0(\{\Gamma^{(2k)}_n\}_{n\ge 0})=\mathcal{T}_0(\{\Gamma_n\}_{n\ge 2k}).\end{aligned} \tag{5.22}$$

From (5.17), (5.18), (5.21), and (5.22) we get

$$\ker D_{\mathcal{T}^*_0}=\mathfrak{H}_1,\ \ker D_{\mathcal{T}_1}=\mathfrak{H}_2,\ldots,\ker D_{\mathcal{T}^*_{2k}}=\mathfrak{H}_{2k+1},\ \ker D_{\mathcal{T}_{2k-1}}=\mathfrak{H}_{2k},\ldots.$$

From (5.12), (5.13), and (5.22) it follows that

$$\begin{aligned}&P_{\ker D_{\mathcal{T}^*_0}}\mathcal{T}_0{\upharpoonright}\ker D_{\mathcal{T}^*_0}=\mathcal{T}_1,\ P_{\ker D_{\mathcal{T}_1}}\mathcal{T}_1{\upharpoonright}\ker D_{\mathcal{T}_1}=\mathcal{T}_2,\ldots,\\ &P_{\ker D_{\mathcal{T}_{2k-1}}}\mathcal{T}_{2k-1}{\upharpoonright}\ker D_{\mathcal{T}_{2k-1}}=\mathcal{T}_{2k},\ P_{\ker D_{\mathcal{T}^*_{2k}}}\mathcal{T}_{2k}{\upharpoonright}\ker D_{\mathcal{T}^*_{2k}}=\mathcal{T}_{2k+1},\ldots.\end{aligned}$$

Thus,

$$\begin{aligned}\mathfrak{H}_{2k-1}&=\ker D_{\mathcal{T}^{*k}_0}\cap\ker D_{\mathcal{T}^{k-1}_0},\\ \mathfrak{H}_{2k}&=\ker D_{\mathcal{T}^{*k}_0}\cap\ker D_{\mathcal{T}^{k}_0}.\end{aligned}$$

In notations of Section 4 the operators $\mathcal{T}_{2k-1}$ and $\mathcal{T}_{2k}$ coincide with the operators $(\mathcal{T}_0)_{k-1,k}$ and $(\mathcal{T}_0)_{k,k}$, respectively. From the definition of $\mathfrak{H}_0$ we get

$$\left(\bigcap_{k\ge 1}\ker D_{\mathcal{T}^{*k}_0}\right)\bigcap\left(\bigcap_{k\ge 1}\ker D_{\mathcal{T}^{k}_0}\right)=\bigcap_{k\ge 1}\left(\ker D_{\mathcal{T}^{*k}_0}\cap\ker D_{\mathcal{T}^{k}_0}\right)=\bigcap_{k\ge 1}\mathfrak{H}_{2k}=\{0\}.$$

So, the operators $\mathcal{T}_0$, $\widetilde{\mathcal{T}}_0$, and $\{\mathcal{T}_k\}_{k\geq 1}$ are completely non-unitary. It follows that the conservative systems

$$\zeta_0 = \left\{ \begin{bmatrix} \Gamma_0 & \mathcal{G}_0 \\ \mathcal{F}_0 & \mathcal{T}_0 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\} \text{ and } \widetilde{\zeta}_0 = \left\{ \begin{bmatrix} \Gamma_0 & \widetilde{\mathcal{G}}_0 \\ \widetilde{\mathcal{F}}_0 & \widetilde{\mathcal{T}}_0 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \widetilde{\mathfrak{H}}_0 \right\}$$

are simple.

The operators $\mathcal{W}_m$ takes the following 2×2 block operator matrix form

$$\mathcal{W}_m = \begin{bmatrix} \Gamma_m & \mathcal{G}_m \\ \mathcal{F}_m & \mathcal{T}_m \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Gamma_{m-1}} \\ \oplus \\ \mathfrak{H}_m \end{array} \to \begin{array}{c} \mathfrak{D}_{\Gamma^*_{m-1}} \\ \oplus \\ \mathfrak{H}_m \end{array},$$

where

$$\begin{aligned}
\mathcal{G}_{2k-1} &= \widetilde{\mathcal{G}}_0(\{\Gamma_n\}_{n\geq 2k-1}) = \begin{bmatrix} D_{\Gamma^*_{2k-1}} & 0 & 0 & \ldots \end{bmatrix} : \mathfrak{H}_{2k-1} \to \mathfrak{D}_{\Gamma^*_{2k-2}}, \\
\mathcal{G}_{2k} &= \mathcal{G}_0(\{\Gamma_n\}_{n\geq 2k}) = \begin{bmatrix} D_{\Gamma^*_{2k}}\Gamma_{2k+1} & D_{\Gamma^*_{2k}}D_{\Gamma^*_{2k+1}} & 0 & 0 & \ldots \end{bmatrix} : \mathfrak{H}_{2k} \to \mathfrak{D}_{\Gamma^*_{2k-1}}, \\
\mathcal{F}_{2k-1} &= \widetilde{\mathcal{F}}_0(\{\Gamma_n\}_{n\geq 2k-1}) = \begin{bmatrix} \Gamma_{2k}D_{\Gamma_{2k-1}} \\ D_{\Gamma_{2k}}D_{\Gamma_{2k-1}} \\ 0 \\ 0 \\ \vdots \end{bmatrix} : \mathfrak{D}_{\Gamma_{2k-2}} \to \mathfrak{H}_{2k-1}, \\
\mathcal{F}_{2k} &= \mathcal{F}_0(\{\Gamma_n\}_{n\geq 2k}) = \begin{bmatrix} D_{\Gamma_{2k}} \\ 0 \\ 0 \\ \vdots \end{bmatrix} : \mathfrak{D}_{\Gamma_{2k-1}} \to \mathfrak{H}_{2k}.
\end{aligned}$$

Suppose that the system

$$\zeta_0 = \left\{ \begin{bmatrix} \Gamma_0 & \mathcal{G}_0 \\ \mathcal{F}_0 & \mathcal{T}_0 \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}$$

has transfer function $\Psi(\lambda)$, i.e.,

$$\Psi(\lambda) = \Gamma_0 + \lambda\mathcal{G}_0(I_{\mathfrak{H}_0} - \lambda\mathcal{T}_0)^{-1}\mathcal{F}_0.$$

Then $\Psi(0) = \Gamma_0$. Let $\Psi_1(\lambda)$ be the first Schur iterate of Ψ. By (4.7) the transfer function of the simple conservative system

$$\Omega_{0,1}(\nu) = \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0}C(D^{-1}_{\Gamma_0}B^*)^* & D^{-1}_{\Gamma^*_0}C\upharpoonright \ker D_{A^*} \\ AP_{\ker D_A}D^{-1}_{A^*}B & P_{\ker D_{A^*}}A\upharpoonright \ker D_{A^*} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_{A^*} \right\}$$

is the first Schur iterate of the transfer function of the simple conservative system

$$\nu = \left\{ \begin{bmatrix} \Gamma_0 & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{N}, \mathfrak{H} \right\}.$$

We will construct the system $\zeta_1 = \Omega_{0,1}(\zeta_0)$ from the system ζ_0. In our case

$$\zeta_1 = \Omega_{0,1}(\zeta_0) = \left\{ \begin{bmatrix} D^{-1}_{\Gamma^*_0}\mathcal{G}_0(D^{-1}_{\Gamma_0}\mathcal{F}^*_0)^* & D^{-1}_{\Gamma^*_0}\mathcal{G}_0\upharpoonright \ker D_{\mathcal{T}^*_0} \\ \mathcal{T}_0P_{\ker D_{\mathcal{T}_0}}D^{-1}_{\mathcal{T}^*_0}\mathcal{F}_0 & P_{\ker D_{\mathcal{T}^*_0}}\mathcal{T}_0\upharpoonright \ker D_{\mathcal{T}^*_0} \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma^*_0}, \ker D_{\mathcal{T}^*_0} \right\}$$

Clearly,

$$D^{-1}_{\Gamma^*_0}\mathcal{G}_0 = \begin{bmatrix}\Gamma_1 & D_{\Gamma^*_1} & 0 & 0 & \ldots\end{bmatrix} : \mathfrak{H}_0 \to \mathfrak{D}_{\Gamma^*_0},$$

$$(D^{-1}_{\Gamma_0}\mathcal{F}^*_0)^* = \begin{bmatrix} I_{\mathfrak{M}} \\ 0 \\ 0 \\ \vdots \end{bmatrix} : \mathfrak{D}_{\Gamma_0} \to \mathfrak{H}_0.$$

Therefore,

$$\begin{bmatrix}\Gamma_1 & D_{\Gamma^*_1} & 0 & 0 & \ldots\end{bmatrix}\begin{bmatrix} I_{\mathfrak{M}} \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \Gamma_1.$$

Thus, the first Schur parameter of Ψ is equal to Γ_1. From (5.17) it follows that $\ker D_{\mathcal{T}^*_0} = \mathfrak{H}_1$ and $\mathfrak{D}_{\mathcal{T}^*_0} = D_{\Gamma_0}P_{\mathfrak{D}_{\Gamma_0}}$. Hence

$$\mathfrak{D}^{-1}_{\mathcal{T}^*_0}\mathcal{F}_0 = \begin{bmatrix} I_{\mathfrak{D}_{\Gamma_0}} \\ 0 \\ 0 \\ \vdots \end{bmatrix} : \mathfrak{D}_{\Gamma_0} \to \mathfrak{H}_0.$$

As has been proved above

$$P_{\ker D_{\mathcal{T}^*_0}}\mathcal{T}_0\restriction \ker D_{\mathcal{T}^*_0} = \mathcal{T}_1.$$

Let $h \in \mathfrak{D}_{\Gamma_0}$. Let us find the projection $P_{\ker D_{\mathcal{T}_0}}h$. According to (5.17) we have to find the vectors $\varphi \in \mathfrak{D}_{\Gamma_1}$ and $\psi \in \mathfrak{D}_{\Gamma^*_0}$ such that

$$\begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} D_{\Gamma_1}\varphi \\ -\Gamma_1\varphi \end{bmatrix} + \begin{bmatrix} \Gamma^*_1\psi \\ D_{\Gamma^*_1}\psi \end{bmatrix}.$$

We have

$$\begin{cases} h = D_{\Gamma_1}\varphi + \Gamma^*_1\psi \\ \Gamma_1\varphi = D_{\Gamma^*_1}\psi. \end{cases}$$

From the second equation and Proposition 2.1 it follows $\varphi = D_{\Gamma_1}g$, $\psi = \Gamma_1 g$, where $g \in \mathfrak{D}_{\Gamma_0}$. Therefore

$$h = D^2_{\Gamma_1}g + \Gamma^*_1\Gamma_1 g,$$

i.e., $g = h$. Hence

$$P_{\ker D_{\mathcal{T}_0}}\mathfrak{D}^{-1}_{\mathcal{T}^*_0}\mathcal{F}_0 h = P_{\ker D_{\mathcal{T}_0}}h = \begin{bmatrix} D^2_{\Gamma_1}h \\ -\Gamma_1 D_{\Gamma_1}h \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Now we get

$$\mathcal{T}_0 P_{\ker D_{\mathcal{T}_0}} \mathfrak{D}^{-1}_{\mathcal{T}_0^*} \mathcal{F}_0 h =$$

$$= \begin{bmatrix} D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^* D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 & D_{\Gamma_2^*} D_{\Gamma_3^*} & 0 & 0 & 0 & \dots & \dots \\ 0 & D_{\Gamma_2} D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 & -\Gamma_2^* D_{\Gamma_3^*} & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \Gamma_4 D_{\Gamma_3} & -\Gamma_4\Gamma_3^* & D_{\Gamma_4^*}\Gamma_5 & D_{\Gamma_4^*} D_{\Gamma_5^*} & 0 & \dots & \dots \\ 0 & 0 & 0 & D_{\Gamma_4} D_{\Gamma_3} & -D_{\Gamma_4}\Gamma_3^* & -\Gamma_4^*\Gamma_5 & -\Gamma_4 D_{\Gamma_5^*} & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \Gamma_6 D_{\Gamma_5} & -\Gamma_6\Gamma_5^* & D_{\Gamma_6^*}\Gamma_7 & D_{\Gamma_6^*} D_{\Gamma_7^*} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \times$$

$$\times \begin{bmatrix} D^2_{\Gamma_1} h \\ -\Gamma_1 D_{\Gamma_1} h \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma_2 D_{\Gamma_1} h \\ D_{\Gamma_2} D_{\Gamma_1} h \\ 0 \\ 0 \\ \vdots \end{bmatrix} \in \mathfrak{H}_1.$$

Thus we get that ζ_1 is of the form

$$\zeta_1 = \Omega_{0,1}(\zeta_0) = \left\{ \begin{bmatrix} \Gamma_1 & \mathcal{G}_1 \\ \mathcal{F}_1 & \mathcal{T}_1 \end{bmatrix}; \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}, \mathfrak{H}_1 \right\} = \left\{ \widetilde{U}_0\left(\{\Gamma_n\}_{n\ge1}\right), \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}, \widetilde{\mathfrak{H}}_0\left(\{\Gamma_n\}_{n\ge1}\right) \right\}.$$

Similarly

$$\Omega_{1,0}(\widetilde{\zeta}_0) = \left\{ U_0\left(\{\Gamma_n\}_{n\ge1}\right), \mathfrak{D}_{\Gamma_0}, \mathfrak{D}_{\Gamma_0^*}, \mathfrak{H}_0\left(\{\Gamma_n\}_{n\ge1}\right) \right\}.$$

The transfer functions of these systems are equal to $\Psi_1(\lambda)$ (see Section 4), and Γ_1 is exactly is the first Schur parameter of $\Psi(\lambda)$.

Let $\Psi_2(\lambda)$ is the second Schur iterate of Ψ. Constructing the simple conservative system $\zeta_2 = \Omega_{1,0}(\zeta_1)$ of the form (4.7) with the transfer function Ψ_2 we will get the system

$$\zeta_2 = \left\{ \begin{bmatrix} \Gamma_2 & \mathcal{G}_2 \\ \mathcal{F}_2 & \mathcal{T}_2 \end{bmatrix}; \mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_1^*}, \mathfrak{H}_2 \right\} = \left\{ \mathcal{U}_0(\{\Gamma_n\}_{n\ge2}); \mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_1^*}, \mathfrak{H}_0(\{\Gamma_n\}_{n\ge2}) \right\}.$$

Let $\Psi_m(\lambda)$ be the $m-$th Schur iterate of Ψ. Arguing by induction we get that $\Psi_m(\lambda)$ is transfer function of the system

$$\zeta_m = \left\{ \begin{bmatrix} \Gamma_m & \mathcal{G}_m \\ \mathcal{F}_m & \mathcal{T}_m \end{bmatrix}; \mathfrak{D}_{\Gamma_{m-1}}, \mathfrak{D}_{\Gamma_{m-1}^*}, \mathfrak{H}_m \right\} =$$
$$= \begin{cases} \left\{ \widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge2k-1}); \mathfrak{D}_{\Gamma_{2k-2}}, \mathfrak{D}_{\Gamma_{2k-2}^*}, \widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\ge2k-1}) \right\}, & m = 2k-1 \\ \left\{ \mathcal{U}_0(\{\Gamma_n\}_{n\ge2k}); \mathfrak{D}_{\Gamma_{2k-1}}, \mathfrak{D}_{\Gamma_{2k-1}^*}, \mathfrak{H}_0(\{\Gamma_n\}_{n\ge2k}) \right\}, & m = 2k \end{cases}$$

for all m. Observe that

$$\zeta_{2k-1} = \Omega_{0,1}(\zeta_{2k-2}),\ \zeta_{2k} = \Omega_{1,0}(\zeta_{2k-1}),\ k \ge 1.$$

Thus, $\{\Gamma_n\}_{n\ge0}$ are the Schur parameters of Ψ. □

From Theorem 5.2 and Theorem 2.2 we immediately arrive at the following result.

Theorem 5.3. *Let $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and let $\{\Gamma_n\}_{n\ge0}$ be the Schur parameters of Θ. Then the systems* (5.19) *are simple conservative realizations of Θ. Moreover, for each natural number*

k the k-th Schur iterate Θ_k of Θ is the transfer function of the simple conservative systems $\left\{\mathcal{U}_0(\Gamma_n\}_{n\ge k});\mathfrak{D}_{\Gamma_{k-1}},\mathfrak{D}_{\Gamma^*_{k-1}},\mathfrak{H}_0(\Gamma_n\}_{n\ge k})\right\}$ *and* $\left\{\widetilde{\mathcal{U}}_0(\Gamma_n\}_{n\ge k});\mathfrak{D}_{\Gamma_{k-1}},\mathfrak{D}_{\Gamma^*_{k-1}},\widetilde{\mathfrak{H}}_0(\Gamma_n\}_{n\ge k})\right\}$.

Observe that in fact we have proved Theorem 1.2 and our proof is different from given in [34] and [20].

Remark 5.4. More complicated construction of the state Hilbert space and simple conservative realization for a Schur function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$ by means of a block operator matrix are given in [47] (see [20]). These constructions also involve Schur parameters of Θ and some additional Hilbert spaces and operators. One more model based on the Schur parameters of a scalar Schur class function Θ is obtained in [41]. In terms of this model in [41] are established the necessary and sufficient conditions in order to Θ has a meromorphic pseudocontinuation of bounded type to the exterior of the unit disk. In recent preprint [43] a construction of a minimal conservative realization of a scalar finite Blaschke product in terms of the Hessenberg matrix is given.

6. Block operator CMV matrices (the rest cases)

Let $\{\Gamma_n\}$ be the Schur parameters of the function $\Theta\in\mathbf{S}(\mathfrak{M},\mathfrak{N})$. Suppose Γ_m is an isometry (respect., co-isometry, unitary) for some $m\ge 0$. Then $\Theta_m(\lambda)=\Gamma_m$ for all $\lambda\in\mathbb{D}$ and

$$\begin{array}{l}\Theta_{m-1}(\lambda)=\Gamma_{m-1}+\lambda D_{\Gamma^*_{m-1}}\Gamma_m(I_{\mathfrak{D}_{\Gamma_{m-1}}}+\lambda\Gamma^*_{m-1}\Gamma_m)^{-1}D_{\Gamma_{m-1}},\\ \Theta_{m-2}(\lambda)=\Gamma_{m-2}+\lambda D_{\Gamma^*_{m-2}}\Theta_{m-1}(\lambda)(I_{\mathfrak{D}_{\Gamma_{m-2}}}+\lambda\Gamma^*_{m-2}\Theta_{m-1}(\lambda))^{-1}D_{\Gamma_{m-2}},\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots,\\ \Theta(\lambda)=\Gamma_0+\lambda D_{\Gamma^*_0}\Theta_1(\lambda)(I_{\mathfrak{D}_{\Gamma_0}}+\lambda\Gamma^*_0\Theta_1(\lambda))^{-1}D_{\Gamma_0},\ \lambda\in\mathbb{D}.\end{array}$$

In this case the function Θ also is the transfer function of the simple conservative systems constructed similarly to the situation in Section 5 by means of its Schur parameters and corresponding block operator CMV matrices $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$. Observe that if Γ_m is isometric (respect., co-isometric) then $\Gamma_n=0$, $\mathfrak{D}_{\Gamma^*_n}=\mathfrak{D}_{\Gamma^*_m}$, $D_{\Gamma^*_n}=I_{\mathfrak{D}_{\Gamma^*_m}}$ (respect., $\mathfrak{D}_{\Gamma_n}=\mathfrak{D}_{\Gamma_m}$, $D_{\Gamma_n}=I_{\mathfrak{D}_{\Gamma_m}}$) for $n>m$. The constructions of the state spaces $\mathfrak{H}_0=\mathfrak{H}_0(\{\Gamma_n\}_{n\ge0})$ and $\widetilde{\mathfrak{H}}_0=\widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\ge0})$ are similar to (5.1) but one have to replace $\mathfrak{D}_{\Gamma_n}$ by $\{0\}$ (respect., $\mathfrak{D}_{\Gamma^*_n}$ by $\{0\}$) for $n\ge m$, and $\mathfrak{D}_{\Gamma^*_n}$ by $\mathfrak{D}_{\Gamma^*_m}$ (respect., $\mathfrak{D}_{\Gamma_n}$ by $\mathfrak{D}_{\Gamma_m}$) for $n>m$. The relation

$$\widetilde{\mathfrak{H}}_0(\{\Gamma^*_n\}_{n\ge0})=\mathfrak{H}_0(\{\Gamma_n\}_{n\ge0})$$

remains true. If, in addition, the operator Γ_m is isometry ($\Longleftrightarrow$ the operator $D_{\Gamma^*_m}$ is the orthogonal projection in $\mathfrak{D}_{\Gamma_{m-1}}$ onto $\ker\Gamma^*_m$) or co-isometry ($\Longleftrightarrow$ the operator D_{Γ_m} is the orthogonal projection in $\mathfrak{D}_{\Gamma^*_{m-1}}$ onto $\ker\Gamma_m$), then the corresponding unitary elementary rotation takes the the row or the column form

$$\mathbf{J}^{(r)}_{\Gamma_m}=\begin{bmatrix}\Gamma_m & I_{\mathfrak{D}_{\Gamma^*_m}}\end{bmatrix}:\begin{array}{c}\mathfrak{D}_{\Gamma_{m-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_m}\end{array}\to\mathfrak{D}_{\Gamma^*_{m-1}},$$

$$\mathbf{J}^{(c)}_{\Gamma_m}=\begin{bmatrix}\Gamma_m\\ D_{\Gamma_m}\end{bmatrix}:\mathfrak{D}_{\Gamma_{m-1}}\to\begin{array}{c}\mathfrak{D}_{\Gamma^*_{m-1}}\\ \oplus\\ \mathfrak{D}_{\Gamma_m}\end{array}.$$

Therefore, in definitions (5.2) of the block diagonal operator matrices

$$\mathcal{L}_0=\mathcal{L}_0(\{\Gamma_n\}_{n\ge0}),\ \mathcal{M}_0=\mathcal{M}_0(\{\Gamma_n\}_{n\ge0}),\ \text{and}\ \widetilde{\mathcal{M}}_0=\widetilde{\mathcal{M}}_0(\{\Gamma_n\}_{n\ge0})$$

one should replace

- $\mathbf{J}_{\Gamma_m}$ by $\mathbf{J}^{(r)}_{\Gamma_m}$ and $\mathbf{J}_{\Gamma_n}$ by $I_{\mathfrak{D}_{\Gamma^*_m}}$ for $n>m$, when Γ_m is isometry,
- $\mathbf{J}_{\Gamma_m}$ by $\mathbf{J}^{(c)}_{\Gamma_m}$, and $\mathbf{J}_{\Gamma_n}$ by $I_{\mathfrak{D}_{\Gamma_m}}$ for $n>m$, when Γ_m is co-isometry,
- $\mathbf{J}_{\Gamma_m}$ by Γ_m, when Γ_m is unitary.

As in Section 5 in all these cases the block operators CMV matrices $\mathcal{U}_0=\mathcal{U}_0(\{\Gamma_n\}_{n\geq 0})$ and $\widetilde{\mathcal{U}}_0=\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\geq 0})$, are given by the products

$$\mathcal{U}_0=\mathcal{L}_0\mathcal{M}_0,\ \widetilde{\mathcal{U}}_0=\widetilde{\mathcal{M}}_0\mathcal{L}_0.$$

These matrices are five block-diagonal. In the case when the operator Γ_m, is unitary the block operator CMV matrices $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ are finite and otherwise they are semi-infinite.

As in Section 5 the truncated block operator CMV matrices $\mathcal{T}_0=\mathcal{T}_0((\{\Gamma_n\}_{n\geq 0})$ and $\widetilde{\mathcal{T}}_0=\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\geq 0})$ are defined by (5.10) and (5.11)

$$\mathcal{T}_0=P_{\mathfrak{H}_0}\mathcal{U}_0\upharpoonright\mathfrak{H}_0,\ \widetilde{\mathcal{T}}_0=P_{\widetilde{\mathfrak{H}}_0}\widetilde{\mathcal{U}}_0\upharpoonright\widetilde{\mathfrak{H}}_0.$$

As before the operators $\mathcal{T}_0$ and $\widetilde{\mathcal{T}}_0$ are unitarily equivalent completely non-unitary contractions and, moreover, the equalities (5.9), (5.14), and Proposition 5.1 hold true. Unlike Section 5 the operators given by truncated block operator CMV matrices $\mathcal{T}_m$ and $\widetilde{\mathcal{T}}_m$ obtaining from $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ by deleting first $m+1$ rows and $m+1$ columns are

- co-shifts of the form
$$\mathcal{T}_m=\widetilde{\mathcal{T}}_m=\begin{bmatrix}0 & I_{\mathfrak{D}_{\Gamma^*_m}} & 0 & 0 & \ldots\\ 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_m}} & 0 & \ldots\\ 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_m}} & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}:\begin{array}{c}\mathfrak{D}_{\Gamma^*_m}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_m}\\ \oplus\\ \vdots\end{array}\to\begin{array}{c}\mathfrak{D}_{\Gamma^*_m}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_m}\\ \oplus\\ \vdots\end{array},$$
when Γ_m is isometry,
- the unilateral shifts of the form
$$\mathcal{T}_m=\widetilde{\mathcal{T}}_m=\begin{bmatrix}0 & 0 & 0 & 0 & \ldots\\ I_{\mathfrak{D}_{\Gamma_m}} & 0 & 0 & 0 & \ldots\\ 0 & I_{\mathfrak{D}_{\Gamma_m}} & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}:\begin{array}{c}\mathfrak{D}_{\Gamma_m}\\ \oplus\\ \mathfrak{D}_{\Gamma_m}\\ \oplus\\ \vdots\end{array}\to\begin{array}{c}\mathfrak{D}_{\Gamma_m}\\ \oplus\\ \mathfrak{D}_{\Gamma_m}\\ \oplus\\ \vdots\end{array},$$
when Γ_m is co-isometry.

One can see that Proposition 5.1 remains true.

Similarly to (5.19) let us consider the conservative systems

$$\zeta_0=\{\mathcal{U}_0;\mathfrak{M},\mathfrak{N},\mathfrak{H}_0\},\ \widetilde{\zeta}_0=\{\widetilde{\mathcal{U}}_0;\mathfrak{M},\mathfrak{N},\widetilde{\mathfrak{H}}_0\}.$$

One can check that the systems ζ_0 and $\widetilde{\zeta}_0$ are simple and unitarily equivalent. Moreover, relations (5.20) and, therefore, Theorem 5.2 and Theorem 5.3 remain valid for a situations considered here.

In order to obtain precise forms of $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ one can consider the following cases:

(1) Γ_{2N} is isometric (co-isometric) for some N,
(2) Γ_{2N+1} is isometric (co-isometric) for some N,

(3) the operator Γ_{2N} is unitary for some N,
(4) the operator Γ_{2N+1} is unitary for some N.

We shall give several examples.

Example 6.1. The operator Γ_4 is isometric. Define the state spaces

$$\mathfrak{H}_0 := \begin{matrix}\mathfrak{D}_{\Gamma_0}\\ \oplus \\ \mathfrak{D}_{\Gamma^*_1}\end{matrix} \bigoplus \begin{matrix}\mathfrak{D}_{\Gamma_2}\\ \oplus \\ \mathfrak{D}_{\Gamma^*_3}\end{matrix} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots,$$

$$\widetilde{\mathfrak{H}}_0 := \begin{matrix}\mathfrak{D}_{\Gamma^*_0}\\ \oplus \\ \mathfrak{D}_{\Gamma_1}\end{matrix} \bigoplus \begin{matrix}\mathfrak{D}_{\Gamma^*_2}\\ \oplus \\ \mathfrak{D}_{\Gamma_3}\end{matrix} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots.$$

Then the spaces $\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0$ and $\mathfrak{N}\bigoplus\mathfrak{H}_0$ can be represented as follows

$$\mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0 = \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_{\Gamma^*_0}\end{matrix} \bigoplus \begin{matrix}\mathfrak{D}_{\Gamma_1}\\ \oplus \\ \mathfrak{D}_{\Gamma^*_2}\end{matrix} \bigoplus \begin{matrix}\mathfrak{D}_{\Gamma_3}\\ \oplus \\ \mathfrak{D}_{\Gamma^*_4}\end{matrix} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots,$$

$$\mathfrak{N}\bigoplus\mathfrak{H}_0 = \begin{matrix}\mathfrak{N}\\ \oplus \\ \mathfrak{D}_{\Gamma_0}\end{matrix} \bigoplus \begin{matrix}\mathfrak{D}_{\Gamma^*_1}\\ \oplus \\ \mathfrak{D}_{\Gamma_2}\end{matrix} \bigoplus \mathfrak{D}_{\Gamma^*_3} \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma^*_4} \bigoplus \ldots.$$

Define the unitary operators

$$\mathcal{M}_0 = I_{\mathfrak{M}} \bigoplus \mathbf{J}_{\Gamma_1} \bigoplus \mathbf{J}_{\Gamma_3} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus \ldots : \mathfrak{M}\bigoplus\mathfrak{H}_0 \to \mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0,$$

$$\widetilde{\mathcal{M}}_0 = I_{\mathfrak{N}} \bigoplus \mathbf{J}_{\Gamma_1} \bigoplus \mathbf{J}_{\Gamma_3} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus \ldots : \mathfrak{N}\bigoplus\mathfrak{H}_0 \to \mathfrak{N}\bigoplus\widetilde{\mathfrak{H}}_0,$$

$$\mathcal{L}_0 = \mathbf{J}_{\Gamma_0} \bigoplus \mathbf{J}_{\Gamma_2} \bigoplus \mathbf{J}^{(r)}_{\Gamma_4} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus I_{\mathfrak{D}_{\Gamma^*_4}} \bigoplus \ldots : \mathfrak{M}\bigoplus\widetilde{\mathfrak{H}}_0 \to \mathfrak{N}\bigoplus\mathfrak{H}_0.$$

Then

$$\mathcal{U}_0 = \mathcal{L}_0\mathcal{M}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma^*_0}\Gamma_1 & D_{\Gamma^*_0}D_{\Gamma^*_1} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_0} & -\Gamma^*_0\Gamma_1 & -\Gamma^*_0 D_{\Gamma^*_1} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma^*_1 & D_{\Gamma^*_2}\Gamma_3 & D_{\Gamma^*_2}D_{\Gamma^*_3} & 0 & 0 & 0 & 0 & \ldots\\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma^*_1 & -\Gamma^*_2\Gamma_3 & -\Gamma^*_2 D_{\Gamma^*_3} & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & \Gamma_4 D_{\Gamma_3} & -\Gamma_4\Gamma^*_3 & I_{\mathfrak{D}_{\Gamma^*_4}} & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_4}} & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_4}} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{M}}_0\mathcal{L}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma^*_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma^*_0 & D_{\Gamma^*_1}\Gamma_2 & D_{\Gamma^*_1}D_{\Gamma^*_2} & 0 & 0 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma^*_0 & -\Gamma^*_1\Gamma_2 & -\Gamma_1 D_{\Gamma^*_2} & 0 & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & \Gamma_3 D_{\Gamma_2} & -\Gamma_3\Gamma^*_2 & D_{\Gamma^*_3}\Gamma_4 & D_{\Gamma^*_3} & 0 & 0 & 0 & \ldots\\ 0 & 0 & D_{\Gamma_3}D_{\Gamma_2} & -D_{\Gamma_3}\Gamma^*_2 & -\Gamma^*_3\Gamma_4 & -\Gamma^*_3 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_4}} & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma^*_4}} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

Example 6.2. The operator Γ_0 is co-isometric. Then

$$\mathfrak{H}_0=\widetilde{\mathfrak{H}}_0=\sum_{n=0}^{\infty}\bigoplus\mathfrak{D}_{\Gamma_0},$$

$$\begin{array}{l}\mathcal{M}_0=I_{\mathfrak{M}}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus\ldots,\\ \widetilde{\mathcal{M}}_0=I_{\mathfrak{N}}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus\ldots,\\ \mathcal{L}_0=\mathbf{J}^{(c)}_{\Gamma_0}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus I_{\mathfrak{D}_{\Gamma_0}}\bigoplus\ldots,\end{array}$$

$$\mathcal{U}_0=\widetilde{\mathcal{U}}_0=\begin{bmatrix}\Gamma_0 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_0} & 0 & 0 & 0 & \ldots\\ 0 & I_{\mathfrak{D}_{\Gamma_0}} & 0 & 0 & \ldots\\ 0 & 0 & I_{\mathfrak{D}_{\Gamma_0}} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

Example 6.3. The operator Γ_2 is co-isometric. In this case

$$\begin{array}{l}\mathfrak{H}_0=\begin{array}{c}\mathfrak{D}_{\Gamma_0}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array}\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\ldots\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\ldots,\\ \widetilde{\mathfrak{H}}_0=\begin{array}{c}\mathfrak{D}_{\Gamma_0^*}\\ \oplus\\ \mathfrak{D}_{\Gamma_1}\end{array}\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\ldots\bigoplus\mathfrak{D}_{\Gamma_2}\bigoplus\ldots,\end{array}$$

$$\mathcal{U}_0=\mathcal{L}_0\mathcal{M}_0=\begin{bmatrix}\mathbf{J}_{\Gamma_0} & & & & \\ & \mathbf{J}^{(c)}_{\Gamma_2} & & & \\ & & I_{\mathfrak{D}_{\Gamma_2}} & & \\ & & & I_{\mathfrak{D}_{\Gamma_2}} & \\ & & & & \ddots\end{bmatrix}\begin{bmatrix}I_{\mathfrak{M}} & & & & \\ & \mathbf{J}_{\Gamma_1} & & & \\ & & I_{\mathfrak{D}_{\Gamma_2}} & & \\ & & & I_{\mathfrak{D}_{\Gamma_2}} & \\ & & & & \ddots\end{bmatrix}=$$

$$=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & 0 & \ldots\\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 & 0 & 0 & \ldots\\ 0 & \Gamma_2D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & 0 & 0 & 0 & \ldots\\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_2}} & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_2}} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix},$$

$$\widetilde{\mathcal U}_0=\widetilde{\mathcal M}_0\mathcal L_0=\begin{bmatrix} I_{\mathfrak M} &&&&\\ & \mathbf J_{\Gamma_1} &&&\\ && I_{\mathfrak D_{\Gamma_2}} &&\\ &&& I_{\mathfrak D_{\Gamma_2}} &\\ &&&&\ddots\end{bmatrix}\begin{bmatrix}\mathbf J_{\Gamma_0}&&&&\\ &\mathbf J^{(c)}_{\Gamma_2}&&&\\ &&I_{\mathfrak D_{\Gamma_2}}&&\\ &&&I_{\mathfrak D_{\Gamma_2}}&\\ &&&&\ddots\end{bmatrix}=$$

$$=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & \ldots\\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & 0 & 0 & 0 & \ldots\\ 0 & 0 & D_{\Gamma_2} & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_2}} & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_2}} & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

Example 6.4. The operator Γ_1 is isometric. In this case

$$\mathfrak H_0=\begin{matrix}\mathfrak D_{\Gamma_0}\\ \oplus\\ \mathfrak D_{\Gamma_1^*}\end{matrix}\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\ldots\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\ldots,$$
$$\widetilde{\mathfrak H}_0=\mathfrak D_{\Gamma_0^*}\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\ldots\bigoplus\mathfrak D_{\Gamma_1^*}\bigoplus\ldots.$$

$$\mathcal U_0=\mathcal L_0\mathcal M_0=\begin{bmatrix}\mathbf J_{\Gamma_0}&&&&\\ &I_{\mathfrak D_{\Gamma_1^*}}&&&\\ &&I_{\mathfrak D_{\Gamma_1^*}}&&\\ &&&I_{\mathfrak D_{\Gamma_1^*}}&\\ &&&&\ddots\end{bmatrix}\begin{bmatrix}I_{\mathfrak M}&&&&\\ &\mathbf J^{(r)}_{\Gamma_1}&&&\\ &&I_{\mathfrak D_{\Gamma_1^*}}&&\\ &&&I_{\mathfrak D_{\Gamma_1^*}}&\\ &&&&\ddots\end{bmatrix}=$$

$$=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^* & 0 & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix},$$

$$\widetilde{\mathcal U}_0=\widetilde{\mathcal M}\mathcal L_0=\begin{bmatrix}I_{\mathfrak N}&&&&\\ &\mathbf J^{(r)}_{\Gamma_1}&&&\\ &&I_{\mathfrak D_{\Gamma_1^*}}&&\\ &&&I_{\mathfrak D_{\Gamma_1^*}}&\\ &&&&\ddots\end{bmatrix}\begin{bmatrix}\mathbf J_{\Gamma_0}&&&&\\ &I_{\mathfrak D_{\Gamma_1^*}}&&&\\ &&I_{\mathfrak D_{\Gamma_1^*}}&&\\ &&&I_{\mathfrak D_{\Gamma_1^*}}&\\ &&&&\ddots\end{bmatrix}=$$

$$=\begin{bmatrix}\Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 0 & I_{\mathfrak D_{\Gamma_1^*}} & 0 & 0 & 0 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{bmatrix}.$$

Example 6.5. The operator Γ_3 is isometric.

$$\mathfrak{H}_0 = \begin{array}{c}\mathfrak{D}_{\Gamma_0}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma_2}\\ \oplus\\ \mathfrak{D}_{\Gamma_3^*}\end{array} \bigoplus \mathfrak{D}_{\Gamma_3^*} \bigoplus \mathfrak{D}_{\Gamma_3^*} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma_3^*} \bigoplus \ldots,$$

$$\widetilde{\mathfrak{H}}_0 = \begin{array}{c}\mathfrak{D}_{\Gamma_0^*}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array} \bigoplus \mathfrak{D}_{\Gamma_2^*} \bigoplus \mathfrak{D}_{\Gamma_3^*} \bigoplus \mathfrak{D}_{\Gamma_3^*} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma_1^*} \bigoplus \ldots.$$

$$\mathcal{U}_0 = \mathcal{L}_0\mathcal{M}_0 = \begin{bmatrix} \mathbf{J}_{\Gamma_0} & & & & & \\ & \mathbf{J}_{\Gamma_2} & & & & \\ & & I_{\mathfrak{D}_{\Gamma_3^*}} & & & \\ & & & I_{\mathfrak{D}_{\Gamma_3^*}} & & \\ & & & & I_{\mathfrak{D}_{\Gamma_3^*}} & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} I_{\mathfrak{M}} & & & & & \\ & \mathbf{J}_{\Gamma_1} & & & & \\ & & \mathbf{J}^{(r)}_{\Gamma_3} & & & \\ & & & I_{\mathfrak{D}_{\Gamma_3^*}} & & \\ & & & & I_{\mathfrak{D}_{\Gamma_3^*}} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots \\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 & D_{\Gamma_2^*} & 0 & 0 & 0 & \ldots \\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 & -\Gamma_2^* & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_3^*}} & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_3^*}} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{M}}_0\mathcal{L}_0 = \begin{bmatrix} I_{\mathfrak{N}} & & & & & \\ & \mathbf{J}_{\Gamma_1} & & & & \\ & & \mathbf{J}^{(r)}_{\Gamma_3} & & & \\ & & & I_{\mathfrak{D}_{\Gamma_3^*}} & & \\ & & & & I_{\mathfrak{D}_{\Gamma_3^*}} & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \mathbf{J}_{\Gamma_0} & & & & & \\ & \mathbf{J}_{\Gamma_2} & & & & \\ & & I_{\mathfrak{D}_{\Gamma_3^*}} & & & \\ & & & I_{\mathfrak{D}_{\Gamma_3^*}} & & \\ & & & & I_{\mathfrak{D}_{\Gamma_3^*}} & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & \ldots \\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & -\Gamma_1^*D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \Gamma_3 D_{\Gamma_2} & -\Gamma_3\Gamma_2^* & I_{\mathfrak{D}_{\Gamma_3^*}} & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_3^*}} & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_3^*}} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Example 6.6. The operator Γ_5 is co-isometric.

$$\mathfrak{H}_0 = \begin{array}{c}\mathfrak{D}_{\Gamma_0}\\ \oplus\\ \mathfrak{D}_{\Gamma_1^*}\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma_2}\\ \oplus\\ \mathfrak{D}_{\Gamma_3^*}\end{array} \bigoplus \mathfrak{D}_{\Gamma_4} \bigoplus \mathfrak{D}_{\Gamma_5} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma_5} \bigoplus \ldots,$$

$$\widetilde{\mathfrak{H}}_0 = \begin{array}{c}\mathfrak{D}_{\Gamma_0^*}\\ \oplus\\ \mathfrak{D}_{\Gamma_1}\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma_2^*}\\ \oplus\\ \mathfrak{D}_{\Gamma_3}\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_{\Gamma_4^*}\\ \oplus\\ \mathfrak{D}_{\Gamma_5}\end{array} \bigoplus \mathfrak{D}_{\Gamma_5} \bigoplus \mathfrak{D}_{\Gamma_5} \bigoplus \ldots \bigoplus \mathfrak{D}_{\Gamma_5} \bigoplus \ldots,$$

$$\begin{aligned}
\mathcal{L}_0 &= \mathbf{J}_{\Gamma_0}\bigoplus\mathbf{J}_{\Gamma_2}\bigoplus\mathbf{J}_{\Gamma_4}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus\ldots,\\
\mathcal{M}_0 &= I_{\mathfrak{M}}\bigoplus\mathbf{J}_{\Gamma_1}\bigoplus\mathbf{J}_{\Gamma_3}\bigoplus\mathbf{J}^{(c)}_{\Gamma_5}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus\ldots,\\
\widetilde{\mathcal{M}}_0 &= I_{\mathfrak{N}}\bigoplus\mathbf{J}_{\Gamma_1}\bigoplus\mathbf{J}_{\Gamma_3}\bigoplus\mathbf{J}^{(c)}_{\Gamma_5}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus I_{\mathfrak{D}_{\Gamma_5}}\bigoplus\ldots.
\end{aligned}$$

$$\mathcal{U}_0=\mathcal{L}_0\mathcal{M}_0=\begin{bmatrix}
\Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\
D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & \Gamma_2D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 & D_{\Gamma_2^*}D_{\Gamma_3^*} & 0 & 0 & 0 & 0 & \ldots\\
0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 & -\Gamma_2^*D_{\Gamma_3^*} & 0 & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & \Gamma_4D_{\Gamma_3} & -\Gamma_4\Gamma_3^* & D_{\Gamma_4^*}\Gamma_5 & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & D_{\Gamma_4}D_{\Gamma_3} & -D_{\Gamma_4}\Gamma_3^* & -\Gamma_4^*\Gamma_5 & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & D_{\Gamma_5} & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_5}} & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_5}} & 0 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0=\widetilde{\mathcal{M}}_0\mathcal{L}_0=\begin{bmatrix}
\Gamma_0 & D_{\Gamma_0^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\\
\Gamma_1D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\
D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & -\Gamma_1^*D_{\Gamma_2^*} & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & 0 & \Gamma_3D_{\Gamma_2} & -\Gamma_3\Gamma_2^* & -D_{\Gamma_3^*}\Gamma_4 & D_{\Gamma_3^*}D_{\Gamma_4^*} & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & \Gamma_4D_{\Gamma_3} & -\Gamma_4\Gamma_3^* & D_{\Gamma_4^*}\Gamma_5 & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & \Gamma_5D_{\Gamma_4} & -\Gamma_5\Gamma_4^* & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & D_{\Gamma_5}D_{\Gamma_4} & -D_{\Gamma_5}\Gamma_4^* & 0 & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_5}} & 0 & 0 & \ldots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{\Gamma_5}} & 0 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$

Example 6.7. The operator Γ_{2N} is unitary. In this case

$$\mathfrak{H}_0=\sum_{n=0}^{N-1}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma^*_{2n+1}}\end{matrix},\quad
\widetilde{\mathfrak{H}}_0=\sum_{n=0}^{N-1}\bigoplus\begin{matrix}\mathfrak{D}_{\Gamma^*_{2n}}\\ \oplus\\ \mathfrak{D}_{\Gamma_{2n+1}}\end{matrix},$$

$$\mathcal{U}_0=\begin{bmatrix}\mathbf{J}_{\Gamma_0} & & & & \\ & \mathbf{J}_{\Gamma_2} & & & \\ & & \ddots & & \\ & & & \mathbf{J}_{\Gamma_{2(N-1)}} & \\ & & & & \Gamma_{2N}\end{bmatrix}\begin{bmatrix}I_{\mathfrak{M}} & & & & \\ & \mathbf{J}_{\Gamma_1} & & & \\ & & \mathbf{J}_{\Gamma_3} & & \\ & & & \ddots & \\ & & & & \mathbf{J}_{\Gamma_{2N-1}}\end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0=\begin{bmatrix}I_{\mathfrak{N}} & & & & \\ & \mathbf{J}_{\Gamma_1} & & & \\ & & \mathbf{J}_{\Gamma_3} & & \\ & & & \ddots & \\ & & & & \mathbf{J}_{\Gamma_{2N-1}}\end{bmatrix}\begin{bmatrix}\mathbf{J}_{\Gamma_0} & & & & \\ & \mathbf{J}_{\Gamma_2} & & & \\ & & \ddots & & \\ & & & \mathbf{J}_{\Gamma_{2(N-1)}} & \\ & & & & \Gamma_{2N}\end{bmatrix}.$$

If $N = 1$ (Γ_2 is unitary) then we have

$$\mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} \\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} \\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* \end{bmatrix}, \ \widetilde{\mathcal{U}}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} & 0 \\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 \\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 \end{bmatrix}$$

Example 6.8. The operator Γ_{2N+1} is unitary.

$$\mathfrak{H}_0 = \mathfrak{D}_{\Gamma_0},\ \widetilde{\mathfrak{H}}_0 = \mathfrak{D}_{\Gamma_0^*} \text{ if } N = 0,$$

$$\mathfrak{H}_0 = \sum_{n=0}^{N-1} \bigoplus \begin{matrix} \mathfrak{D}_{\Gamma_{2n}} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n+1}^*} \end{matrix} \bigoplus \mathfrak{D}_{\Gamma_{2N}},\ \widetilde{\mathfrak{H}}_0 = \sum_{n=0}^{N-1} \bigoplus \begin{matrix} \mathfrak{D}_{\Gamma_{2n}^*} \\ \oplus \\ \mathfrak{D}_{\Gamma_{2n+1}} \end{matrix} \bigoplus \mathfrak{D}_{\Gamma_{2N}^*} \text{ if } N \ge 1,$$

$$\mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} \\ D_{\Gamma_0} & -\Gamma_0^* \end{bmatrix} \begin{bmatrix} I_{\mathfrak{M}} & 0 \\ 0 & \Gamma_1 \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*}\Gamma_1 \\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 \end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \begin{bmatrix} I_{\mathfrak{N}} & 0 \\ 0 & \Gamma_1 \end{bmatrix} \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} \\ D_{\Gamma_0} & -\Gamma_0^* \end{bmatrix} = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} \\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* \end{bmatrix}, \text{ if } N = 0,$$

$$\mathcal{U}_0 = \begin{bmatrix} \mathbf{J}_{\Gamma_0} & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \ddots & \\ & & & \mathbf{J}_{\Gamma_{2N}} \end{bmatrix} \begin{bmatrix} I_{\mathfrak{M}} & & & & & \\ & \mathbf{J}_{\Gamma_1} & & & & \\ & & \mathbf{J}_{\Gamma_3} & & & \\ & & & \ddots & & \\ & & & & \mathbf{J}_{\Gamma_{2N-1}} & \\ & & & & & \Gamma_{2N+1} \end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \begin{bmatrix} I_{\mathfrak{N}} & & & & & \\ & \mathbf{J}_{\Gamma_1} & & & & \\ & & \mathbf{J}_{\Gamma_3} & & & \\ & & & \ddots & & \\ & & & & \mathbf{J}_{\Gamma_{2N-1}} & \\ & & & & & \Gamma_{2N+1} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{\Gamma_0} & & & \\ & \mathbf{J}_{\Gamma_2} & & \\ & & \ddots & \\ & & & \mathbf{J}_{\Gamma_{2N}} \end{bmatrix}, \text{ if } N \ge 1.$$

If $N = 1$ (Γ_3 is unitary) then

$$\mathcal{U}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*}\Gamma_1 & D_{\Gamma_0^*}D_{\Gamma_1^*} & 0 \\ D_{\Gamma_0} & -\Gamma_0^*\Gamma_1 & -\Gamma_0^*D_{\Gamma_1^*} & 0 \\ 0 & \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^* & D_{\Gamma_2^*}\Gamma_3 \\ 0 & D_{\Gamma_2}D_{\Gamma_1} & -D_{\Gamma_2}\Gamma_1^* & -\Gamma_2^*\Gamma_3 \end{bmatrix}, \ \widetilde{\mathcal{U}}_0 = \begin{bmatrix} \Gamma_0 & D_{\Gamma_0^*} & 0 & 0 \\ \Gamma_1 D_{\Gamma_0} & -\Gamma_1\Gamma_0^* & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*} \\ D_{\Gamma_1}D_{\Gamma_0} & -D_{\Gamma_1}\Gamma_0^* & -\Gamma_1^*\Gamma_2 & -\Gamma_1^*D_{\Gamma_2^*} \\ 0 & 0 & \Gamma_3 D_{\Gamma_2} & -\Gamma_3\Gamma_2^* \end{bmatrix}.$$

7. Unitary operators with cyclic subspaces, dilations, and block operator CMV matrices

7.1. **Carathéodory class functions associated with conservative systems.**

Definition 7.1. *Let $\mathfrak{M}$ be a separable Hilbert space. The class $\mathbf{C}(\mathfrak{M})$ of $\mathbf{L}(\mathfrak{M})$-valued functions holomorphic on the unit disk $\mathbb{D}$ and having positive real part for all $\lambda \in \mathbb{D}$ is called the Carathéodory class.*

Consider a conservative systems $\tau = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{H} \right\}$ whose input and output spaces coincide. Put

$$\mathcal{H} = \mathfrak{M} \oplus \mathfrak{H}$$

and let the function $F_\tau(z)$ be defined as follows

$$F_\tau(\lambda)=P_{\mathfrak M}(U_\tau+\lambda I_{\mathcal H})(U_\tau-\lambda I_{\mathcal H})^{-1}\upharpoonright\mathfrak M,\ \lambda\in\mathbb D, \tag{7.1}$$

where

$$U_\tau=\begin{bmatrix}D & C\\ B & A\end{bmatrix}:\begin{array}{c}\mathfrak M\\ \oplus\\ \mathfrak H\end{array}\to\begin{array}{c}\mathfrak M\\ \oplus\\ \mathfrak H\end{array}$$

is unitary operator in $\mathcal H$ associated with the system τ. The function $F_\tau(z)$ is holomorphic in $\mathbb D$ and

$$F_\tau(\lambda)+F_\tau^*(\lambda)=2(1-|\lambda|^2)P_{\mathfrak M}(U_\tau^*-\bar\lambda I_{\mathcal H})^{-1}(U_\tau-\lambda I_{\mathcal H})^{-1}\upharpoonright\mathfrak M.$$

It follows that $F_\tau(\lambda)+F_\tau^*(\lambda)\ge 0$ for all $\lambda\in\mathbb D$.

The function $F_\tau(\lambda)$ defined by (7.1) belongs to the Carathéodory class $\mathbf C(\mathfrak M)$ and, in addition, $F_\tau(0)=I_{\mathfrak M}$. We also shall consider the function

$$\widetilde F_\tau(\lambda):=F_\tau^*(\bar\lambda)=P_{\mathfrak M}(I_{\mathcal H}+\lambda U_\tau)(I_{\mathcal H}-\lambda U_\tau)^{-1}.$$

The functions F_τ and $\widetilde F_\tau$ we will call the Carathéodory functions associated with conservative system $\tau=\left\{\begin{bmatrix}D & C\\ B & A\end{bmatrix};\mathfrak M,\mathfrak M,\mathfrak H\right\}$.

Proposition 7.2. *Let*

$$\tau=\left\{\begin{bmatrix}D & C\\ B & A\end{bmatrix};\mathfrak M,\mathfrak M,\mathfrak H\right\}$$

be a conservative system. Then the transfer function $\Theta_\tau(\lambda)$ and the Carathéodory function $F_\tau(\lambda)$ are connected by the following relations

$$\begin{aligned}&\Theta_\tau^*(\bar\lambda)=\frac{1}{\lambda}(F_\tau(\lambda)-I_{\mathfrak M})(F_\tau(\lambda)+I_{\mathfrak M})^{-1},\\ &F_\tau(\lambda)=(I_{\mathfrak M}+\lambda\Theta_\tau^*(\bar\lambda))(I_{\mathfrak M}-\lambda\Theta_\tau^*(\bar\lambda))^{-1},\ \lambda\in\mathbb D.\end{aligned} \tag{7.2}$$

Proof. We use the well known Schur–Frobenius formula for the inverse of block operators. Let Φ be a bounded linear operator given by the block operator matrix

$$\Phi=\begin{pmatrix}X & Y\\ Z & W\end{pmatrix}:\begin{array}{c}\mathfrak M\\ \oplus\\ \mathfrak H\end{array}\to\begin{array}{c}\mathfrak M\\ \oplus\\ \mathfrak H\end{array}.$$

Suppose that $W^{-1}\in\mathbf L(\mathfrak H)$ and $(X-YW^{-1}Z)^{-1}\in\mathbf L(\mathfrak M)$. Then $\Phi^{-1}\in\mathbf L(\mathfrak M\oplus\mathfrak H,\mathfrak M\oplus\mathfrak H)$ and

$$\Phi^{-1}=\begin{pmatrix}K^{-1} & -K^{-1}YW^{-1}\\ -W^{-1}ZK^{-1} & W^{-1}+W^{-1}ZK^{-1}YW^{-1}\end{pmatrix},$$

where $K=X-YW^{-1}Z$. Applying this formula for

$$\Phi=I_{\mathcal H}-\lambda U_\tau=\begin{pmatrix}I_{\mathfrak M}-\lambda D & -\lambda C\\ -\lambda B & I_{\mathfrak H}-\lambda A\end{pmatrix},\quad\lambda\in\mathbb D,$$

we get $K=I_{\mathfrak M}-\lambda D-\lambda^2C(I_{\mathfrak H}-\lambda A)^{-1}B=I_{\mathfrak M}-\lambda\Theta_\tau(\lambda)$. Therefore

$$P_{\mathfrak M}(I_{\mathcal H}-\lambda U_\tau)^{-1}\upharpoonright\mathfrak M=(I_{\mathfrak M}-\lambda\Theta_\tau(\lambda))^{-1},\quad\lambda\in\mathbb D.$$

Hence

$$P_{\mathfrak M}(I_{\mathcal H}-\lambda U_\tau^*)^{-1}\upharpoonright\mathfrak M=(I_{\mathfrak M}-\lambda\Theta_\tau^*(\bar\lambda))^{-1},\quad\lambda\in\mathbb D.$$

Since U_τ is unitary, from (7.1) we get

$$\begin{aligned}&F_\tau(\lambda) = P_{\mathfrak{M}}(I_{\mathcal{H}} + \lambda U_\tau^*)(I_{\mathcal{H}} - \lambda U_\tau^*)^{-1}\restriction\mathfrak{M} = \\ &= -I_{\mathfrak{M}} + 2P_{\mathfrak{M}}(I_{\mathcal{H}} - \lambda U_\tau^*)^{-1}\restriction\mathfrak{M} = -I_{\mathfrak{M}} + 2(I_{\mathfrak{M}} - \lambda\Theta_\tau^*(\bar{\lambda}))^{-1} = \\ &= (I_{\mathfrak{M}} + \lambda\Theta_\tau^*(\bar{\lambda}))(I_{\mathfrak{M}} - \lambda\Theta_\tau^*(\bar{\lambda}))^{-1}, \quad \lambda \in \mathbb{D}.\end{aligned}$$

□

The following theorem is well known (see [29]).

Theorem 7.3. *Let $\mathfrak{M}$ be a separable Hilbert space and let $F(\lambda) \in \mathbf{C}(\mathfrak{M})$. Then*

(1) *$F(\lambda)$ admits the integral representation*

$$F(\lambda) = \frac{1}{2}(F(0) - F^*(0)) + \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda}\, d\Sigma(t),$$

where $\Sigma(t)$ is a non-decreasing and nonnegative $\mathbf{L}(\mathfrak{M})$-valued function on $[0, 2\pi]$;

(2) *under the condition $F(0) = I_{\mathfrak{M}}$ there exists a Hilbert space $\mathcal{H}$ containing $\mathfrak{M}$ as a subspace, and a unitary operator U in $\mathcal{H}$ such that*

$$F(\lambda) = P_{\mathfrak{M}}(U + \lambda I_{\mathcal{H}})(U - \lambda I_{\mathcal{H}})^{-1}\restriction\mathfrak{M};$$

moreover, the pair $\{\mathcal{H}, U\}$ can be chosen minimal in the sense

$$\overline{\operatorname{span}}\,\{U^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathcal{H}.$$

Proposition 7.4. [41]. *The conservative system*

$$\tau = \left\{\begin{bmatrix} D & C \\ B & A\end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{H}\right\}$$

is simple if and only if

$$\overline{\operatorname{span}}\,\{U_\tau^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathcal{H}.$$

Proof. Let τ be a simple conservative system. Suppose $h \in \mathcal{H}$ and h is orthogonal to $U_\tau^n\mathfrak{M}$ for all $n \in \mathbb{Z}$. Then the vectors $U_\tau^{*n}h$ are orthogonal to $\mathfrak{M}$ in $\mathcal{H}$ for all $n \in \mathbb{Z}$. It follows that $h \in \mathfrak{H}$ and

$$\begin{aligned}&Ch = CAh = CA^2h = \ldots = CA^nh = \ldots = 0,\\ &B^*h = B^*A^*h = B^*A^{*2}h = \ldots B^*A^{*n}h = \ldots = 0.\end{aligned} \tag{7.3}$$

Hence $h \in \left(\bigcap_{n\geq 0}\ker(CA^n)\right) \cap \left(\bigcap_{n\geq 0}\ker(B^*A^{*n})\right)$. Since τ is simple we get $h = 0$, i.e.,

$$\overline{\operatorname{span}}\,\{U_\tau^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathcal{H}.$$

Conversely, let $\overline{\operatorname{span}}\,\{U_\tau^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathcal{H}$. Suppose thar relations (7.3) hold for some $h \in \mathfrak{H}$. Then $h \perp U_\tau^n\mathfrak{M}$ for all $n \in \mathbb{Z}$. Hence $h = 0$ and τ is simple. □

7.2. **Unitary operators with cyclic subspaces.** Let U be a unitary operator in a separable Hilbert space $\mathfrak{K}$ and let $\mathfrak{M}$ be a subspace of $\mathfrak{K}$. Put $\mathfrak{H} = \mathfrak{K} \ominus \mathfrak{M}$. Then U takes the block operator matrix form

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix}.$$

Since U is unitary, the system

$$\eta = \left\{ \begin{bmatrix} D & C \\ B & A \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{H} \right\}$$

is conservative. By Proposition 7.4 the system η is simple if and only if

$$\overline{\operatorname{span}}\,\{U^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathfrak{K}. \tag{7.4}$$

A subspace $\mathfrak{M}$ of $\mathfrak{K}$ is called *cyclic* for U if the condition (7.4) is satisfied.

Define the Carathéodory function

$$F_{\mathfrak{M}}(\lambda) = P_{\mathfrak{M}}(U + \lambda I_{\mathcal{H}})(U - \lambda I_{\mathcal{H}})^{-1}{\upharpoonright}\,\mathfrak{M},\ \lambda \in \mathbb{D}$$

and a Schur function

$$E_{\mathfrak{M}}(\lambda) = \frac{1}{\lambda}(F_{\mathfrak{M}}(\lambda) - I_{\mathfrak{M}})(F_{\mathfrak{M}}(\lambda) + I_{\mathfrak{M}})^{-1},\ \lambda \in \mathbb{D}.$$

According to Proposition 7.2 the transfer function $\Theta(\lambda)$ of the system η and the function $E_{\mathfrak{M}}(\lambda)$ are connected by the relation

$$\Theta(\lambda) = E^*_{\mathfrak{M}}(\bar\lambda),\ \lambda \in \mathbb{D}.$$

Theorem 7.5. *Let U be a unitary operator in a separable Hilbert space and let $\mathfrak{M}$ be a cyclic subspace for U. Then U is unitarily equivalent to the block operator CMV matrices $\mathcal{U}_0(\{\Gamma_n\}_{n\ge0})$ and $\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge0})$ in the Hilbert spaces $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{H}_0(\{\Gamma_n\}_{n\ge0})$ and $\widetilde{\mathcal{H}} = \mathfrak{M} \oplus \widetilde{\mathfrak{H}}_0(\{\Gamma_n\}_{n\ge0})$, respectively, where $\{\Gamma_n\}_{n\ge0}$ are the Schur parameters of the function*

$$\Theta(\lambda) = \frac{1}{\lambda}(F^*_{\mathfrak{M}}(\bar\lambda) - I_{\mathfrak{M}})(F^*_{\mathfrak{M}}(\bar\lambda) + I_{\mathfrak{M}})^{-1}.$$

Proof. Because $\mathfrak{M}$ is a cyclic subspace for U, the conservative system η is simple. By Theorem 5.3 the system η is unitarily equivalent to the systems ζ_0 and $\widetilde{\zeta}_0$ given by (5.19). From (3.5) it follows that U is unitarily equivalent to $\mathcal{U}_0(\{\Gamma_n\}_{n\ge0})$ and $\widetilde{\mathcal{U}}_0(\{\Gamma_n\}_{n\ge0})$. $\square$

Suppose that the cyclic subspace $\mathfrak{M}$ for unitary operator U in $\mathfrak{K}$ is one-dimensional. Let $\varphi \in \mathfrak{M}$, $||\varphi|| = 1$, and let $\mu(\zeta) = (\mathcal{E}(\zeta)\varphi, \varphi)_{\mathfrak{K}}$, where $\mathcal{E}(\zeta)$, $\zeta \in \mathbb{T}$, is the resolution of the identity for U. Then the scalar Carathéodory function $F(\lambda)$ is of the form

$$F(\lambda) = \big((U + \lambda I_{\mathcal{H}})(U - \lambda I_{\mathcal{H}})^{-1}\varphi, \varphi\big)_{\mathfrak{K}} = \int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda}\, d\mu(\zeta)\,, \quad \lambda \in \mathbb{D}.$$

Thus, the function $F(\lambda)$ is associated with the probability measure μ on $\mathbb{T}$. The Schur function associated with μ [60] is the function

$$E(\lambda) = \frac{1}{\lambda}\frac{F(\lambda) - 1}{F(\lambda) + 1},\ \lambda \in \mathbb{D}.$$

By Geronimus theorem [44] the Schur parameters of the function $E(\lambda)$ coincide with Verblunsky coefficient $\{\alpha_n\}_{n\ge 0}$ of the measure μ (see [60]). Let $\Theta(\lambda) := \overline{E(\bar\lambda)}$, $\lambda \in \mathbb{D}$ and let $\{\gamma_n\}_{n\ge 0}$ be the Schur parameters of Θ. Then $\bar\alpha_n = \gamma_n$ for all n and the CMV matrices $\mathcal{U}_0 = \mathcal{U}_0(\{\gamma_n\}_{n\ge 0})$ and $\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0(\{\gamma_n\}_{n\ge 0})$ coincide with the CMV matrices $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ given by (1.2) and (1.3), correspondingly. Observe that $\dim \mathfrak{K} = m \iff$ the function $E(\lambda)$ is the Blaschke product of the form

$$E(\lambda) = e^{i\varphi}\prod_{k=1}^{m}\frac{\lambda-\lambda_k}{1-\bar\lambda_k\lambda}.$$

7.3. **Unitary dilations of a contraction.** Let T be a contraction acting in a Hilbert space H. The unitary operator U in a Hilbert space $\mathcal{H}$ containing H as a subspace is called the unitary dilation of T if $T^n = P_H U^n$ for all $n \in \mathbb{N}$ [64]. Two unitary dilations U in $\mathcal{H}$ and U' in $\mathcal{H}'$ of T are called isomorphic if there exists a unitary operator $W \in \mathbf{L}(\mathcal{H}, \mathcal{H}')$ such that

$$W\restriction H = I_H \quad \text{and} \quad WU = U'W.$$

It is established in [64] that for every contraction T in the Hilbert space H there exists a unitary dilation U in a space H such that U is *is minimal* [64], i.e.,

$$\overline{\mathrm{span}}\,\{U^n H,\ n \in \mathbb{Z}\} = \mathcal{H}.$$

Moreover, two minimal unitary dilations of T are isomorphic [64]. The minimal unitary dilation by means of the infinite matrix form is constructed in [64] on the base of Schäffer paper [55]. Below we show that the minimal unitary dilations can be given by the operator CMV matrices.

Theorem 7.6. *Let T be a contraction in a Hilbert space H. Define the Hilbert spaces*

$$(7.5)\qquad \begin{array}{l} \mathfrak{H}_0 = \begin{array}{c}\mathfrak{D}_T\\ \oplus\\ \mathfrak{D}_{T^*}\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_T\\ \oplus\\ \mathfrak{D}_{T^*}\end{array} \bigoplus \cdots, \\ \widetilde{\mathfrak{H}}_0 = \begin{array}{c}\mathfrak{D}_{T^*}\\ \oplus\\ \mathfrak{D}_T\end{array} \bigoplus \begin{array}{c}\mathfrak{D}_{T^*}\\ \oplus\\ \mathfrak{D}_T\end{array} \bigoplus \cdots, \end{array}$$

and the Hilbert spaces $\mathcal{H}_0 = H \oplus \mathfrak{H}_0$, and $\widetilde{\mathcal{H}}_0 = H \oplus \widetilde{\mathfrak{H}}_0$. Let

$$\mathbf{J}_0 = \begin{bmatrix} 0 & I_{\mathfrak{D}_{T^*}} \\ I_{\mathfrak{D}_T} & 0 \end{bmatrix} : \begin{array}{c}\mathfrak{D}_T\\ \oplus\\ \mathfrak{D}_{T^*}\end{array} \to \begin{array}{c}\mathfrak{D}_{T^*}\\ \oplus\\ \mathfrak{D}_T\end{array}$$

Define operators

$$(7.6)\qquad \begin{array}{l} \mathcal{M}_0 = I_H \bigoplus \mathbf{J}_0 \bigoplus \mathbf{J}_0 \bigoplus \cdots : \mathcal{H}_0 \to \widetilde{\mathcal{H}}_0, \\ \mathcal{L}_0 = \mathbf{J}_T \bigoplus \mathbf{J}_0 \bigoplus \mathbf{J}_0 \bigoplus \cdots : \widetilde{\mathcal{H}}_0 \to \mathcal{H}_0, \end{array}$$

and

$$(7.7)\qquad \mathcal{U}_0 = \mathcal{L}_0\mathcal{M}_0 : \mathcal{H}_0 \to \mathcal{H}_0,\ \widetilde{\mathcal{U}}_0 = \mathcal{M}_0\mathcal{L}_0 : \widetilde{\mathcal{H}}_0 \to \widetilde{\mathcal{H}}_0.$$

Then $\{\mathcal{H}_0, \mathcal{U}_0\}$ and $\{\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{U}}_0\}$ are unitarily equivalent minimal unitary dilations of the operator T.

Proof. Define the $\mathbf{L}(H)$-valued function

$$\widetilde{F}(\lambda) = (I_H + \lambda T)(I_H - \lambda T)^{-1},\ \lambda \in \mathbb{D}.$$

Then the function $\widetilde{F}$ belongs to the Carathéodory class $\mathbf{C}(H)$ and

$$\Theta(\lambda) = T = \frac{1}{\lambda}(\widetilde{F}(\lambda) - I_H)(\widetilde{F}(\lambda) + I_H)^{-1},\ \lambda \in \mathbb{D},$$

belongs to the Schur class $\mathbf{S}(H,H)$. The Schur parameters of Θ is the sequence

$$\Gamma_0 = T, \Gamma_n = 0 \in \mathbf{S}(\mathfrak{D}_T, \mathfrak{D}_{T^*}),\ n \in \mathbb{N}.$$

Let $\mathfrak{H}_0$ and $\widetilde{\mathfrak{H}}_0$ be defined by (7.5), $\mathcal{H}_0 = H \oplus \mathfrak{H}_0$, $\widetilde{\mathcal{H}}_0 = H \oplus \widetilde{\mathfrak{H}}_0$. Then the operators $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$ defined by (7.7) are the block operator CMV matrices constructed by means of the Schur parameters of Θ. Let $\zeta_0 = \{\mathcal{H}_0, \mathfrak{M}, \mathfrak{M}, \mathfrak{H}_0\}$ and $\widetilde{\zeta}_0 = \{\widetilde{\mathcal{U}}_0, \mathfrak{M}, \mathfrak{M}, \widetilde{\mathfrak{H}}_0\}$ be the corresponding conservative systems. By Theorem 5.3 the systems ζ_0 and $\widetilde{\zeta}_0$ are simple, unitary equivalent, and their transfer functions are equal Θ. By Proposition 7.2 we have

$$\begin{aligned}&(I_H + \lambda T)(I_H - \lambda T)^{-1} = \widetilde{F}(\lambda) = (I_H + \lambda\Theta(\lambda))(I_H - \lambda\Theta(\lambda))^{-1} = \\ &= P_H(I_{\mathcal{H}_0} + \lambda\mathcal{U}_0)(I_{\mathcal{H}_0} - \lambda\mathcal{U}_0)^{-1}\upharpoonright H.\end{aligned}$$

Hence

$$T^n = P_H\mathcal{U}_0^n\upharpoonright H,\ n = 0, 1, \ldots.$$

Therefore $\mathcal{U}_0$ is a unitary dilation of T in $\mathcal{H}_0$. By Proposition 7.4 this dilation is minimal. Similarly the operator $\widetilde{\mathcal{U}}_0$ is a minimal unitary dilation of T in $\widetilde{\mathcal{H}}_0$. $\square$

Taking into account (7.6), (5.6), and (5.7) we obtain the following operator matrix forms for minimal unitary dilations $\mathcal{U}_0$ and $\widetilde{\mathcal{U}}_0$:

$$\mathcal{U}_0 = \begin{bmatrix} T & 0 & D_{T^*} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ D_T & 0 & -T^* & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & 0 & 0 & 0 & 0 & \ldots \\ 0 & I_{\mathfrak{D}_T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & 0 & 0 & \ldots \\ 0 & 0 & 0 & I_{\mathfrak{D}_T} & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$\widetilde{\mathcal{U}}_0 = \begin{bmatrix} T & D_{T^*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & 0 & 0 & 0 & 0 & 0 & \ldots \\ D_T & -T^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & 0 & 0 & 0 & \ldots \\ 0 & 0 & I_{\mathfrak{D}_T} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathfrak{D}_{T^*}} & 0 & \ldots \\ 0 & 0 & 0 & 0 & I_{\mathfrak{D}_T} & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

7.4. **The Naimark dilation.** Let $\mathfrak{M}$ be a separable Hilbert space. Denote by $\mathfrak{B}(\mathbb{T})$ the σ-algebra of Borelian subsets of the unit circle $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$. Let μ be a $\mathbf{L}(\mathfrak{M})$-valued Borel measure on $\mathfrak{B}(\mathbb{T})$, i.e.,

(a) for any $\delta \in \mathfrak{B}(\mathbb{T})$ the operator $\mu(\delta)$ is nonnegative,
(b) $\mu(\emptyset) = 0$,
(c) μ is σ-additive with respect to the strong operator convergence.

Denote by $\mathbf{M}(\mathbb{T}, \mathfrak{M})$ the set of all $\mathbf{L}(\mathfrak{M})$-valued Borel measures.

Definition 7.7. [33], [20], [41]. *Let $\mu \in \mathbf{M}(\mathbb{T}, \mathfrak{M})$ be a probability measure ($\mu(\mathbb{T}) = I_{\mathfrak{M}}$) and let the operators $\{S_n\}_{n\in\mathbb{Z}}$ be the sequence of Fourier coefficients of μ, i.e.,*

$$S_n = \int\limits_{\mathbb{T}} \xi^{-n}\mu(d\xi),\ n \in \mathbb{Z}.$$

A Naimark dilation of μ is a pair $\{\mathcal{H}, \mathcal{U}\}$, where $\mathcal{H}$ is a separable Hilbert space containing $\mathfrak{M}$ as a subspace, $\mathcal{U}$ is unitary operator in $\mathcal{H}$ such that

$$S_n = P_{\mathfrak{M}}\mathcal{U}^n\restriction\mathfrak{M},\ n \in \mathbb{Z}.$$

A Naimark dilation is called minimal if

$$\overline{\operatorname{span}}\,\{\mathcal{U}^n\mathfrak{M},\ n \in \mathbb{Z}\} = \mathcal{H}.$$

Proposition 7.8. [33], [20], [41]. *Let $\{\mathcal{H}_1, \mathcal{U}_1\}$ and $\{\mathcal{H}_2, \mathcal{U}_2\}$ be two minimal Naimark dilations of a probability measure $\mu \in \mathbf{M}(\mathbb{T}, \mathfrak{M})$. Then there exists a unitary operator $\mathcal{W} \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that*

$$\mathcal{W}\mathcal{U}_1 = \mathcal{U}_2\mathcal{W} \quad \text{and} \quad \mathcal{W}\restriction\mathfrak{M} = I_{\mathfrak{M}}.$$

The minimal Naimark dilation is constructed by T. Constantinescu in [33] by means of the infinite in both sides block operator matrix whose entries depend on some choice sequence. Here we construct the minimal Naimark dilations in the form of block operator CMV matrices.

Theorem 7.9. *Let $\mathfrak{M}$ be a separable Hilbert space and let $\mu \in \mathbf{M}(\mathbb{T}, \mathfrak{M})$ be a probability measure. Define the functions*

$$F(\lambda) = \int\limits_{\mathbb{T}} \frac{\xi + \lambda}{\xi - \lambda}\,\mu(d\xi),\ \lambda \in \mathbb{D},$$

$$E(\lambda) = \frac{1}{\lambda}\,(F(\lambda) - I_{\mathfrak{M}})(F(\lambda) + I_{\mathfrak{M}})^{-1}.$$

Then $E(\lambda)$ belongs to the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{M})$. Let $\{G_n\}_{n\ge 0}$ be the Schur parameters of E. Construct the Hilbert spaces

$$\mathfrak{H}_0 = \mathfrak{H}_0(\{G_n\}_{n\ge0}),\ \widetilde{\mathfrak{H}}_0 = \widetilde{\mathfrak{H}}_0(\{G_n\}_{n\ge0})$$

and the Hilbert spaces

$$\mathcal{H}_0 = \mathfrak{M} \oplus \mathfrak{H}_0,\ \widetilde{\mathcal{H}}_0 = \mathfrak{M} \oplus \widetilde{\mathfrak{H}}_0.$$

Let

$$\mathcal{U}_0 = \mathcal{U}_0(\{G_n\}_{n\ge0}),\ \widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0(\{G_n\}_{n\ge0})$$

be the block operator CMV matrices constructing by means of $\{G_n\}$. Then the pairs $\{\mathcal{H}_0, \mathcal{U}_0\}$ and $\{\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{U}}_0\}$ are unitarily equivalent minimal Naimark dilations of the measure μ.

Proof. The function $F(\lambda)$ has the Taylor expansion

$$F(\lambda)=I_{\mathfrak M}+2\sum_{n=1}^{\infty}\lambda^n\int\limits_{\mathbb T}\xi^{-n}\mu(d\xi)=I_{\mathfrak M}+2\sum_{n=1}^{\infty}\lambda^nS_n.$$

Then

$$F^*(\bar\lambda)=I_{\mathfrak M}+2\sum_{n=1}^{\infty}\lambda^nS_{-n}.$$

Because $F(\lambda)+F^*(\lambda)\ge 0$ for $\lambda\in\mathbb D$, the $\mathbf L(\mathfrak M)$-valued function $E(\lambda)$ belongs to the Schur class $\mathbf S(\mathfrak M,\mathfrak M)$. Construct the Hilbert spaces $\mathfrak H_0=\mathfrak H_0(\{G_n\}_{n\ge0})$, $\mathcal H_0=\mathfrak M\oplus\mathfrak H_0$ and let $\mathcal U_0=\mathcal U_0(\{G_n\}_{n\ge0})=(\mathcal U_0(\{G_n\}_{n\ge0}))^*$ be the block operator CMV matrix. Then $\mathcal U_0$ is unitary operator in the Hilbert space $\mathcal H_0$. The system $\zeta_0=\{\mathcal U_0;\mathfrak M,\mathfrak M,\mathfrak H_0\}$ is a conservative and simple, and its transfer function is equal to $E(\lambda)$ (see Subsection 5.3, (5.19), Theorem 5.3). Hence the transfer of the adjoint system $\zeta_0^*=\{\mathcal U_0^*;\mathfrak M,\mathfrak M,\mathfrak H_0\}$ is equal to $\Theta(\lambda)=E^*(\bar\lambda)$. By definition of $F(\lambda)$ and $E(\lambda)$, and by Proposition 7.2, and (7.2) we have

$$\begin{aligned}F(\lambda)&=(I_{\mathfrak M}+\lambda E(\lambda))(I_{\mathfrak M}-\lambda E(\lambda))^{-1}=(I_{\mathfrak M}+\lambda\Theta^*(\bar\lambda))(I_{\mathfrak M}-\lambda\Theta^*(\bar\lambda))^{-1}=\\&=P_{\mathfrak M}(\mathcal U_0^*+\lambda I_{\mathcal H_0})(\mathcal U_0^*-\lambda I_{\mathcal H_0})^{-1}\restriction\mathfrak M.\end{aligned}$$

Hence

$$\begin{aligned}F(\lambda)&=I_{\mathfrak M}+2\sum_{n=1}^{\infty}\lambda^nP_{\mathfrak M}\mathcal U_0^n\restriction\mathfrak M,\\F^*(\bar\lambda)&=I_{\mathfrak M}+2\sum_{n=1}^{\infty}\lambda^nP_{\mathfrak M}\mathcal U_0^{-n}\restriction\mathfrak M.\end{aligned}$$

Thus, the pair $\{\mathcal H_0,\mathcal U_0\}$ is the minimal Naimark dilation of the measure μ. The same is true for the pair $\{\widetilde{\mathcal H}_0,\widetilde{\mathcal U}_0\}$. □

8. The block operator CMV matrix models for completely non-unitary contractions

Theorem 8.1. *Let T be a completely non-unitary contraction in a separable Hilbert space H. Let*

$$\Phi_T(\lambda)=(-T+\lambda D_{T^*}(I_H-\lambda T^*)^{-1}D_T)\restriction\mathfrak D_T$$

be the Sz.-Nagy–Foias characteristic function of T [64]. If $\{\Gamma_n\}_{n\ge0}$ are the Schur parameters of $\Phi_T(\lambda)$, then the operator T is unitarily equivalent to the truncated block operator CMV matrices $\mathcal T_0(\{\Gamma_n^*\}_{n\ge0})$ and $\widetilde{\mathcal T}_0(\{\Gamma_n^*\}_{n\ge0})$.

Proof. Consider the simple conservative system

$$\eta=\left\{\begin{bmatrix}-T^*&D_T\\D_{T^*}&T\end{bmatrix};\mathfrak D_{T^*},\mathfrak D_T,H\right\}.$$

The transfer function of η is given by

$$\Theta_\eta(\lambda)=\left(-T^*+\lambda D_T(I_H-\lambda T)^{-1}D_{T^*}\right)\restriction\mathfrak D_{T^*},\ \lambda\in\mathbb D.$$

Since

$$\Phi_T(\lambda)=\left(-T+\lambda D_{T^*}(I_H-\lambda T^*)^{-1}D_T\right)\restriction\mathfrak D_T,\ \lambda\in\mathbb D,$$

we get $\Phi_T(\lambda)=\Theta_\eta^*(\bar\lambda)$, $\lambda\in\mathbb D$. Hence, if $\{\Gamma_n\}_{n\ge0}$ are the Schur parameters of $\Phi_T(\lambda)$, then $\{\Gamma_n^*\}_{n\ge0}$ are the Schur parameters of $\Theta_\eta(\lambda)$. Construct the Hilbert spaces $\mathfrak H_0=\mathfrak H_0(\{\Gamma_n^*\}_{n\ge0})$,

$\widetilde{\mathfrak{H}}_0 = \widetilde{\mathfrak{H}}_0(\{\Gamma^*_n\}_{n\geq 0})$, the block operator CMV matrices $\mathcal{U}_0 = \mathcal{U}_0(\{\Gamma^*_n\}_{n\geq 0})$, $\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0(\{\Gamma^*_n\}_{n\geq 0})$, truncated block CMV matrices $\mathcal{T}_0 = \mathcal{T}_0(\{\Gamma^*_n\}_{n\geq 0})$ and $\widetilde{\mathcal{T}}_0 = \widetilde{\mathcal{T}}_0(\{\Gamma^*_n\}_{n\geq 0})$. Consider the corresponding conservative systems

$$\zeta_0 = \{\mathcal{U}_0; \mathfrak{D}_{T^*}, \mathfrak{D}_T, \mathfrak{H}_0\}, \ \widetilde{\zeta}_0 = \left\{\widetilde{\mathcal{U}}_0; \mathfrak{D}_{T^*}, \mathfrak{D}_T, \widetilde{\mathfrak{H}}_0\right\}.$$

By Theorem 5.3 the systems ζ_0 and $\widetilde{\zeta}_0$ are simple conservative realizations of the function Θ. It follows that the operator T is unitarily equivalent to the operators $\mathcal{T}_0(\{\Gamma^*_n\}_{n\geq 0})$ and $\widetilde{\mathcal{T}}_0(\{\Gamma^*_n\}_{n\geq 0})$. □

Observe that $\mathcal{T}_0(\{\Gamma^*_n\}_{n\geq 0}) = \left(\widetilde{\mathcal{T}}_0(\{\Gamma_n\}_{n\geq 0})\right)^*$ and $\widetilde{\mathcal{T}}_0(\{\Gamma^*_n\}_{n\geq 0}) = (\mathcal{T}_0(\{\Gamma_n\}_{n\geq 0}))^*$.

The results of Sz.-Nagy–Foias [64, Theorem VI.3.1] states that if the function $\Theta \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ is purely contractive ($||\Theta(0)f|| < ||f||$ for all $f \in \mathfrak{M} \setminus \{0\}$) then there exists a completely non-unitary contraction T whose characteristic function coincides with Θ. Here we give another proof of this result.

Theorem 8.2. *Let the function $\Theta(\lambda) \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ be purely contractive. If $\{\Gamma_n\}_{n\geq 0}$ are the Schur parameters of $\Theta(\lambda)$ then the characteristic functions of completely non-unitary contractions given by truncated block operator CMV matrices $\mathcal{T}_0(\{\Gamma^*_n\}_{n\geq 0})$ and $\widetilde{\mathcal{T}}_0(\{\Gamma^*_n\}_{n\geq 0})$ coincide with Θ.*

Proof. Let $\widetilde{\Theta}(\lambda) := \Theta^*(\bar{\lambda})$. Then $\{\Gamma^*_n\}_{n\geq 0}$ are the Schur parameters of $\widetilde{\Theta}$. Construct the Hilbert spaces $\mathfrak{H}_0 = \mathfrak{H}_0(\{\Gamma^*_n\}_{n\geq 0})$, $\widetilde{\mathfrak{H}}_0 = \widetilde{\mathfrak{H}}_0(\{\Gamma^*_n\}_{n\geq 0})$, the block operator CMV matrices $\mathcal{U}_0 = \mathcal{U}_0(\{\Gamma^*_n\}_{n\geq 0})$, $\widetilde{\mathcal{U}}_0 = \widetilde{\mathcal{U}}_0(\{\Gamma^*_n\}_{n\geq 0})$, truncated block CMV matrices $\mathcal{T}_0 = \mathcal{T}_0(\{\Gamma^*_n\}_{n\geq 0})$, $\widetilde{\mathcal{T}}_0 = \widetilde{\mathcal{T}}_0(\{\Gamma^*_n\}_{n\geq 0})$, and consider the corresponding conservative systems

$$\zeta_0 = \{\mathcal{U}_0; \mathfrak{M}, \mathfrak{N}, \mathfrak{H}_0\}, \ \widetilde{\zeta}_0 = \left\{\widetilde{\mathcal{U}}_0; \mathfrak{M}, \mathfrak{N}, \widetilde{\mathfrak{H}}_0\right\}.$$

Then the transfer functions of ζ_0 and $\widetilde{\zeta}_0$ are equal to $\widetilde{\Theta}(\lambda)$. Since the operator

$$\mathcal{U}_0 = \begin{bmatrix} \Gamma^*_0 & \mathcal{G}_0 \\ \mathcal{F}_0 & \mathcal{T}_0 \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{H}_0 \end{array} \to \begin{array}{c} \mathfrak{N} \\ \oplus \\ \mathfrak{H}_0 \end{array}$$

is a contraction, there exist contractions (see [19], [38], [59]) $\mathcal{K} \in \mathbf{L}(\mathfrak{D}_{\mathcal{T}_0}, \mathfrak{N})$, $\mathcal{M} \in \mathbf{L}(\mathfrak{M}, \mathfrak{H}_0)$, $\mathcal{X} \in \mathbf{L}(\mathfrak{D}_{\mathcal{M}}, \mathfrak{D}_{\mathcal{K}^*})$ such that

$$\mathcal{G}_0 = \mathcal{K} D_{\mathcal{T}_0}, \ \mathcal{F}_0 = \mathcal{D}_{\mathcal{T}^*_0}\mathcal{M}, \ \Gamma^*_0 = -\mathcal{K}\mathcal{T}^*_0\mathcal{M} + D_{\mathcal{K}^*}\mathcal{X} D_{\mathcal{M}}.$$

Because $\mathcal{U}_0$ is unitary, the operators $\mathcal{K}$, $\mathcal{M}^*$ are isometries and $\mathcal{X}$ is unitary (see [8],[9]). The characteristic function of $\mathcal{T}^*_0$ and the transfer function of the system ζ_0 are connected by the relation (see [10], [9])

$$\widetilde{\Theta}(\lambda) = \mathcal{K}\Phi_{\mathcal{T}^*_0}(\lambda)\mathcal{M} + \mathcal{X} D_{\mathcal{M}}.$$

Because the operator $D_{\mathcal{M}}$ is the orthogonal projection in $\mathfrak{M}$ onto $\ker \mathcal{M}$, and

$$\widetilde{\Theta}(\lambda){\upharpoonright} \ker \mathcal{M} = \mathcal{X},$$

we have for $f \in \ker \mathcal{M}$

$$||\Gamma^*_0 f|| = ||\widetilde{\Theta}(0)f|| = ||\mathcal{X} f|| = ||f||.$$

Since Γ_0^* is a pure contraction, we obtain $\ker \mathcal{M} = \{0\}$. Similarly $\ker \mathcal{K}^* = \{0\}$, i.e., $\mathcal{K}$ and $\mathcal{M}$ are unitary operators, and $\widetilde{\Theta}(\lambda) = \mathcal{K}\Phi_{\mathcal{T}_0^*}(\lambda)\mathcal{M}$, $\lambda \in \mathbb{D}$. Thus the characteristic function $\Phi_{\mathcal{T}_0}$ of $\mathcal{T}_0$ coincides with Θ. Similarly, the characteristic function $\Phi_{\widetilde{\mathcal{T}}_0}$ of $\widetilde{\mathcal{T}}_0$ coincides with Θ. □

Remark 8.3. For completely non-unitary contractions with one-dimensional defect operators and for a scalar Schur class functions Theorem 8.1 and Theorem 8.2 have been established in [12].

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